■ 1. Find the left-hand limit $\lim_{x\to 2^-} \frac{|x-2|}{x-2}$.

[✓ A] -1
B 1
C -2
D 2

Solution: **A**

If we try substitution to evaluate the limit, we get the undefined value $0/0$. Instead, let's try substituting a value to the left of $x=2$ that's very close to $x=2$, like $x=1.9999$.

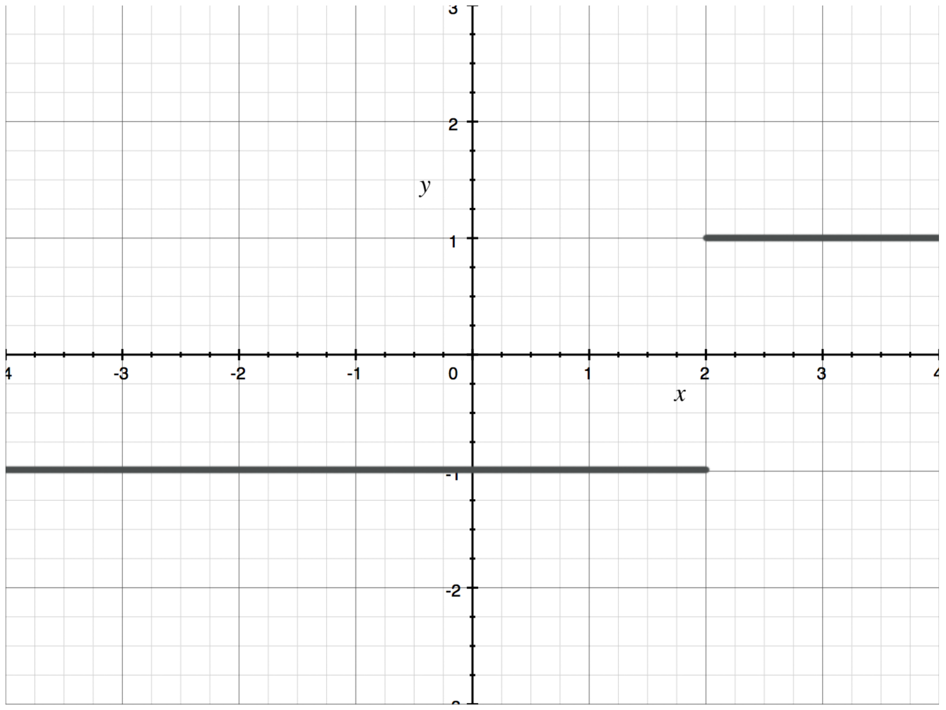
$$\frac{|1.9999-2|}{1.9999-2}$$

$$\frac{|-0.0001|}{-0.0001}$$

$$\frac{0.0001}{-0.0001}$$

$$-1$$

As we approach $x=2$ from the left, the function is a constant -1 (the numerator is always positive and the denominator is always negative). The graph of the function confirms this value for the left-hand limit.



■ **2. If $f(x) = \begin{cases} \ln x & 0 < x \le 2 \\ x^2 \ln 2 & 2 < x \le 4 \end{cases}$, then $\lim_{x\to 2^+} f(x)$ is**

A $\ln 2$

B $\ln 8$

$\boxed{\text{C}}$ $\ln 16$

D 4

Solution: C

When evaluating one-sided limits, we need to decide which piece of the piecewise function we'll substitute into. In this case, we substitute into $x^2 \ln 2$, since we're coming from the positive side of 2.

$$\lim_{x\to 2^+} f(x) = 2^2 \ln 2$$

Applying laws of logarithms gives

$$2^2 \ln 2 = 4 \ln 2 = \ln(4^2) = \ln 16$$

■ **3. Evaluate the limit $\lim_{x\to 3}\frac{x^2-7x+12}{x^2-9}$.**

A $\frac{1}{3}$

B $-\frac{1}{3}$

C $\frac{1}{6}$

$\boxed{\text{D}}$ $-\frac{1}{6}$

Solution: D

Factor the numerator and denominator as completely as possible.

$$\lim_{x\to 3}\frac{x^2-7x+12}{x^2-9}$$

$$\lim_{x\to 3}\frac{(x-4)(x-3)}{(x+3)(x-3)}$$

Cancel the common factor of $x-3$.

$$\lim_{x\to 3}\frac{x-4}{x+3}$$

Then use direct substitution to evaluate the limit.

$$\frac{3-4}{3+3}$$

$$-\frac{1}{6}$$

■ 4. Evaluate the limit $\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3}$.

$\boxed{\text{A}}$ 6

B 3

C 9

D 0

Solution: A

The conjugate of $\sqrt{x}-3$ is $\sqrt{x}+3$. Multiply both the numerator and denominator by this conjugate.

$$\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3}\left(\frac{\sqrt{x}+3}{\sqrt{x}+3}\right)$$

$$\lim_{x\to 9} \frac{(x-9)(\sqrt{x}+3)}{(\sqrt{x}-3)(\sqrt{x}+3)}$$

$$\lim_{x\to 9} \frac{(x-9)(\sqrt{x}+3)}{x+3\sqrt{x}-3\sqrt{x}-9}$$

$$\lim_{x\to 9} \frac{(x-9)(\sqrt{x}+3)}{x-9}$$

Cancel the common factor of $x-9$ from both the numerator and denominator.

$$\lim_{x\to 9} \sqrt{x}+3$$

Now use substitution to evaluate the limit.

$$\sqrt{9}+3$$

$$3+3$$

$$6$$

■ **5.** $\lim_{x\to\infty}\frac{(2x-1)(3-x)}{(x-1)(x+3)}$

A -3

[✓ B] -2

C 2

D 3

Solution: B

This problem is essentially a horizontal asymptote problem, since we're looking for the limit as $x \to \infty$. Rewrite the numerator and denominator by multiplying binomials.

$$\lim_{x\to\infty}\frac{6x-2x^2-3+x}{x^2+3x-x-3}$$

$$\lim_{x\to\infty}\frac{-2x^2+7x-3}{x^2+2x-3}$$

Because the numerator and denominator have equal degree, the horizontal asymptote is given by the ratio of coefficients of the highest-degree terms.

$$\frac{-2}{1}$$

$$-2$$

■ 6. For $x \geq 0$, the horizontal line $y = 2$ is an asymptote for the graph of the function f. Which of the following statements is true?

A $f(0) = 2$

B $f(x) \neq 2$ for all x

C $\lim_{x \to 2} f(x) = \infty$

$\boxed{\text{D}}$ $\lim_{x \to \infty} f(x) = 2$

Solution: D

Given that $y = 2$ is the horizontal asymptote for $x \geq 0$, we know that the limit as $x \to \infty$ must be 2.

B isn't a correct answer because it's false to say that a function can't cross its horizontal asymptote. For example, the function $f(x) = \dfrac{4x+2}{x^2+4x-5}$ crosses its horizontal asymptote.

■ 7. Given $\lim_{x \to 3^-} f(x) = \infty$, which of the following statements must be true?

A $\lim_{x \to 3^+} f(x) = \infty$

B $f(3)$ is undefined

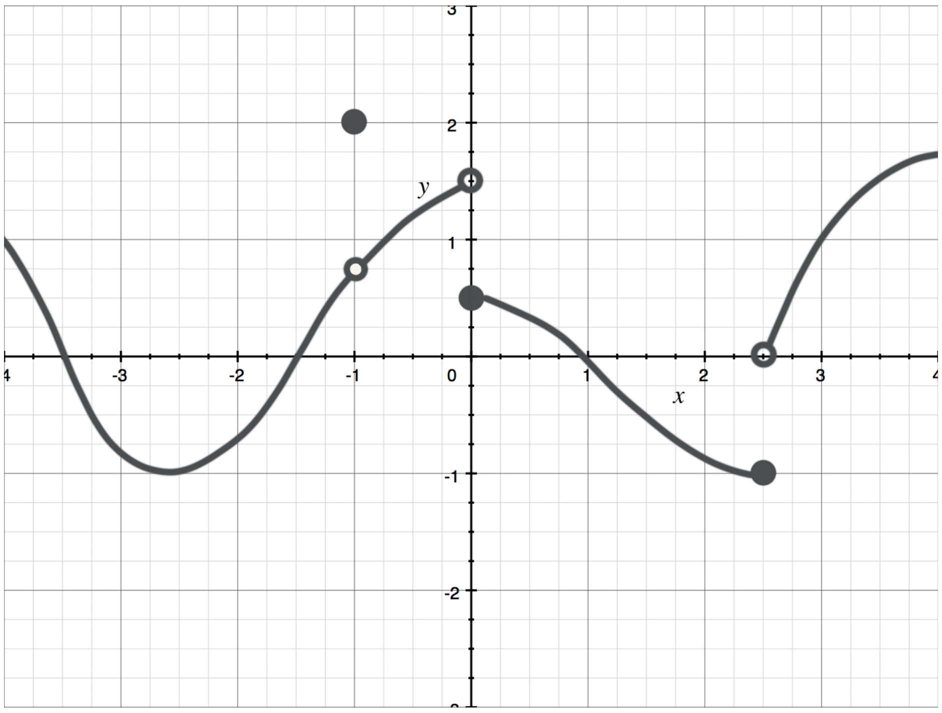
C $f(x)$ has a vertical asymptote when $x = -3$

D $f(x)$ has a vertical asymptote when $x = 3$

Solution: D

In this case, we're looking at the definition of a vertical asymptote. If a function approaches either ∞ or $-\infty$ as we approach a given x-value from either side (in this case $x = 3$ from the left) we have a vertical asymptote.

■ 8. Use the graph to find the function's limit as $x \to 0^-$ and $x \to 0^+$.



A $\lim_{x\to 0^-} f(x) = -\frac{1}{2}$ $\lim_{x\to 0^+} f(x) = -\frac{3}{2}$

B $\lim_{x\to 0^-} f(x) = -\frac{3}{2}$ $\lim_{x\to 0^+} f(x) = -\frac{1}{2}$

C $\lim_{x\to 0^-} f(x) = \frac{1}{2}$ $\lim_{x\to 0^+} f(x) = \frac{3}{2}$

[✓ D] $\lim_{x\to 0^-} f(x)=\frac{3}{2}$ $\lim_{x\to 0^+} f(x)=\frac{1}{2}$

Solution: D

Using the graph, we'll look at the limit as x gets close to 0 from the left side. We can see that

$$\lim_{x\to 0^-} f(x)=\frac{3}{2}$$

And as x gets close to 0 from the right side, we can see that

$$\lim_{x\to 0^+} f(x)=\frac{1}{2}$$

■ 9. Which of the following statements is true?

I. $\lim_{x\to\infty}\frac{2^x}{x^{22}}=0$

II. $\lim_{x\to\infty}\frac{2^x}{\ln x}=0$

III. $\lim_{x\to\infty}\frac{\ln x}{x^2}=0$

[✓ A] III only

B I and II

C I only

D None are true

Solution: A

Think about the problem as a race to infinity. If we're looking at the limit as $x \to \infty$ of a function of the form $h(x) = \frac{f(x)}{g(x)}$, then the limit will be 0 if the denominator grows significantly faster than the numerator. The limit will be ∞ if the numerator grows faster than the denominator. The following inequality holds in the long run with regards to rate of growth:

Logarithms < Polynomials < Exponentials

So in this case, only statement III is correct. In statements I and II, the numerator grows faster than the denominator, so the value of the limit will be ∞, not 0.

■ 10. Given that $f(x) = -x^2$, $g(x) = x^2$, and that $f(x) \le h(x) \le g(x)$ for all values of x, which of the following statements is true?

A $\lim_{x\to\infty} h(x) = 0$

$\boxed{\text{B}}$ $\lim_{x\to 0} h(x) = 0$

C $\lim_{x\to\infty} h(x) = \infty$

D $\lim_{x\to 0} h(x) = -1$

Solution: B

This is the Squeeze Theorem. Both $f(x)$ and $g(x)$ approach 0 as $x \to 0$, and $h(x)$ is squeezed between them. Therefore, the limit as $x \to 0$ of $h(x)$ also must be 0.

■ 11. Which of the following statements would not guarantee that $f(x)$ is continuous at $x=3$?

[✓ A] $\lim_{x\to 3} f(x) = \lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x)$

B $f(3) = \lim_{x\to 3} f(x)$

C $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x) = f(3)$

D $f(x)$ is a polynomial

Solution: A

This is the definition of continuity. In order for a function to be continuous at a point $x=a$, we must prove that the left- and right-hand limits exist at a, that those one-sided limits are equal to each other, and that the value of the function at a is equivalent to the value of the one-sided limits at a.

The statement in answer choice C includes all three of those parts, so it guarantees continuity. Polynomial functions are continuous everywhere, so the statement in answer choice D guarantees continuity. The statement in answer choice B indicates that the general limit exists, which means that the one-sided limits both exist and are equal, so this statement is equivalent to the statement in answer choice C, and therefore also guarantees continuity.

Answer choice A does not guarantee continuity, because it doesn't rule out the possibility that there's a point discontinuity at $x=3$.

■ 12. Given the table below showing select values of a continuous function. Which of the following statements must be true?

x	1	3	7	9	10
f(x)	-3	2	4	-2	2

I. $f(x) > 0$ on the interval $(3,7)$

II. $f(x) = 0$ for at least one x value on the interval $(7,9)$

III. $f(x)$ has exactly 3 zeros on the interval $(1,10)$

[✓ A] II only

B II and III

C None are true

D All are true

Solution: A

Statement I is not necessarily true because, while the average rate of change on $(3,7)$ is positive, there's no guarantee that the function doesn't dip down below the x-axis on that interval.

Statement II is true by the Intermediate Value Theorem.

Statement III is false. Though the minimum number of zeros on the interval $(1,10)$ would be 3, there's nothing to guarantee that there are not more zeros in the interval.

■ 13. Determine the value of c that makes $f(x) = \begin{cases} x+3 & x \le -1 \\ 2x-c & x > -1 \end{cases}$ continuous for all real numbers.

Solution:

For the function to be continuous, the pieces of the function must be equivalent when $x = -1$.

$$x + 3 = 2x - c$$

$$-1 + 3 = 2(-1) - c$$

$$2 = -2 - c$$

$$4 = -c$$

$$c = -4$$

■ 14. Draw a diagram of a function on the given set of axes that meets all of the requirements below.

a. $\lim_{x\to a^-} = 2$

b. $\lim_{x\to a^+} = -2$

c. $f(a) = -2$

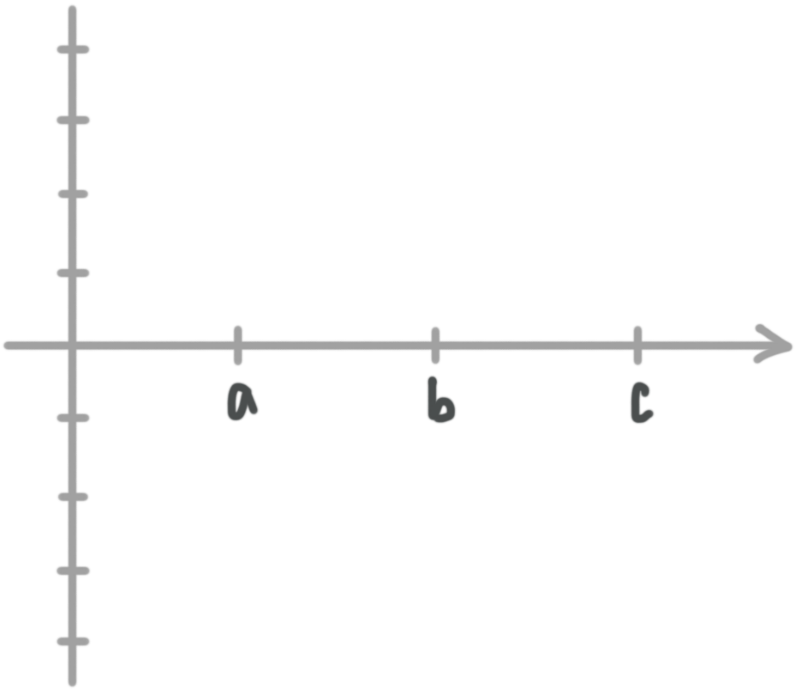
d. $f(b) =$ DNE

e. $\lim_{x\to b} = -2$

f. $\lim_{x\to c^-} = \infty$

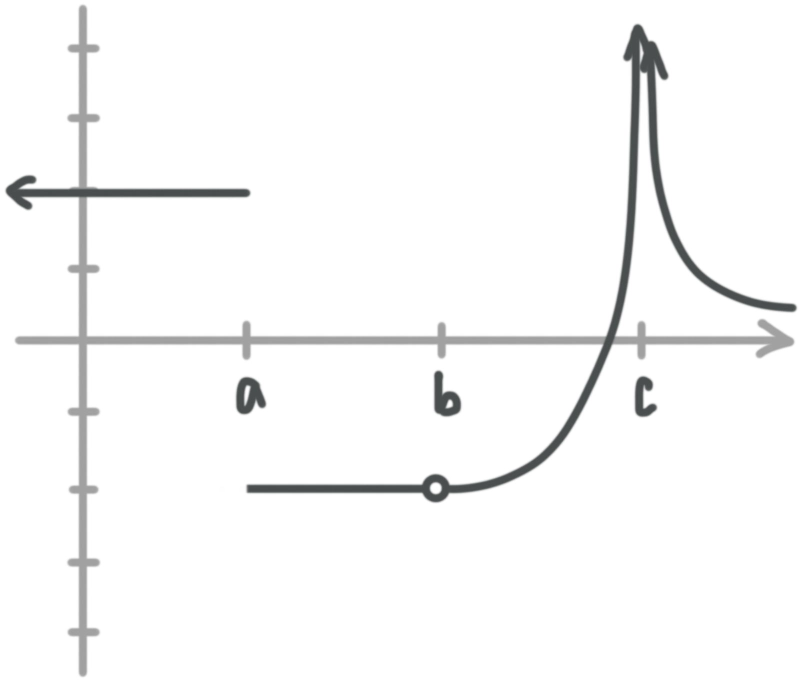
g. $\lim_{x\to c^+} = \infty$

h. $\lim_{x\to\infty} = 0$



Solution:

There are infinitely many possibilities for this problem, and one solution is given below:



■ 15. $\lim_{x\to 2}\frac{x^3-4x^2-4x+16}{x-2}$

Solution:

Direct substitution gives an indeterminate form. Use long division of polynomials to rewrite the rational function. (L'Hospital's Rule would also work here.)

$$\lim_{x\to 2} x^2-2x-8$$

With the function rewritten, use direct substitution.

$$2^2-2(2)-8$$

$$4-4-8$$

$$-8$$

■ **16.** $\lim_{x\to 0}\frac{2-\sqrt{x+4}}{x}$

Solution:

Direct substitution gives an indeterminate form. Apply conjugate method.

$$\lim_{x\to 0}\frac{2-\sqrt{x+4}}{x}\left(\frac{2+\sqrt{x+4}}{2+\sqrt{x+4}}\right)$$

$$\lim_{x\to 0}\frac{4+2\sqrt{x+4}-2\sqrt{x+4}-(x+4)}{x(2+\sqrt{x+4})}$$

$$\lim_{x\to 0}\frac{4-x-4}{x(2+\sqrt{x+4})}$$

$$\lim_{x\to 0}\frac{-x}{x(2+\sqrt{x+4})}$$

Now we can cancel an x from the numerator and denominator and use direct substitution.

$$\lim_{x\to 0}-\frac{1}{2+\sqrt{x+4}}$$

$$-\frac{1}{2+\sqrt{0+4}}$$

$$-\frac{1}{2+2}$$

$$-\frac{1}{4}$$

■ **17.** $\lim_{x\to 0}\frac{\frac{4}{x+3}-\frac{4}{3}}{x}$

Solution:

Direct substitution gives an indeterminate form. Rewrite the function by clearing the fractions in the numerator.

$$\lim_{x\to 0}\frac{\frac{4}{x+3}-\frac{4}{3}}{x}\left(\frac{3(x+3)}{3(x+3)}\right)$$

$$\lim_{x\to 0}\frac{\frac{12(x+3)}{x+3}-\frac{12(x+3)}{3}}{3x(x+3)}$$

$$\lim_{x\to 0}\frac{12-4(x+3)}{3x(x+3)}$$

$$\lim_{x\to 0}\frac{12-4x-12}{3x(x+3)}$$

Now we can cancel the x term and use direct substitution.

$$\lim_{x\to 0}\frac{-4x}{3x(x+3)}$$

$$\lim_{x \to 0} -\frac{4}{3(x+3)}$$

$$-\frac{4}{3(0+3)}$$

$$-\frac{4}{9}$$