■ 1. Given that $f(x)$ is a continuous and differentiable function for all real numbers, which of the following would give the instantaneous rate of change of $f(x)$ when $x=4$?

I. $f'(4)$

II. $\lim_{h\to 0}\frac{f(4+h)-f(4)}{h}$

III. $\lim_{x\to 4}\frac{f(x)-f(4)}{x-4}$

A I only

B I and II

C II and III

[✓ D] I, II, and III

Solution: D

We need to make the link that the instantaneous rate of change of a function is the derivative of the function at $x=4$.

Statement I is true because it's proper notation for the derivative of f at the point $x=4$. Statement II is true because this is the limit definition of the derivative for the function $f(x)$ when $x=4$. Statement III is true because this is the first step in using the alternate form of the definition of the derivative. Therefore, all three statements are true.

■ 2. $\lim_{h\to 0}\frac{\cos(\pi+h)+1}{h}$

[✓ A] 0

B 1

C -1

D Does not exist

Solution: A

The expression is in the form of the limit definition of the derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The $\cos(\pi + h)$ term tells us that we're looking at the derivative of the function $f(x) = \cos x$, evaluated at π. We can confirm this by realizing that $f(\pi) = -1$, which is why we're seeing the $+1$ in the numerator.

To find $f'(\pi)$, we'll take the derivative of $f(x) = \cos x$,

$$f'(x) = -\sin x$$

and then evaluate at $x = \pi$.

$$f'(\pi) = -\sin \pi$$

$$f'(\pi) = 0$$

■ 3. Let f be the function defined as $f(x) = x^{x+1}$. Select values of $f(x)$ are given in the table below. If the values in the table are used to approximate $f(2.5)$, what is the difference between the estimate and the actual value?

x	2	3
f(x)	8	81

A 0

B -12.724

[✓ C] 48.295

D 15.775

Solution: C

First use the difference quotient and the values in the table to find an estimate of $f(2.5)$.

$$f(2.5) \approx \frac{f(3)-f(2)}{3-2} = \frac{81-8}{1} = 73$$

Then use a calculator to find the exact value value of $f(2.5)$.

$$f(2.5) = 2.5^{2.5+1}$$

$$f(2.5) = 2.5^{3.5}$$

$$f(2.5) \approx 24.705$$

So the difference between the estimated value and the actual value is

$$73 - 24.705 \approx 48.295$$

■ 4. Which of the following statements cannot be true?

A A function is non-differentiable at $x = c$ and the limit from the left and right at $x = c$ are equal.

[✓ B] A function is discontinuous and differentiable at a point $x = c$.

C A function is continuous and non-differentiable at a point $x = c$

D A function is continuous and differentiable at a point $x = c$.

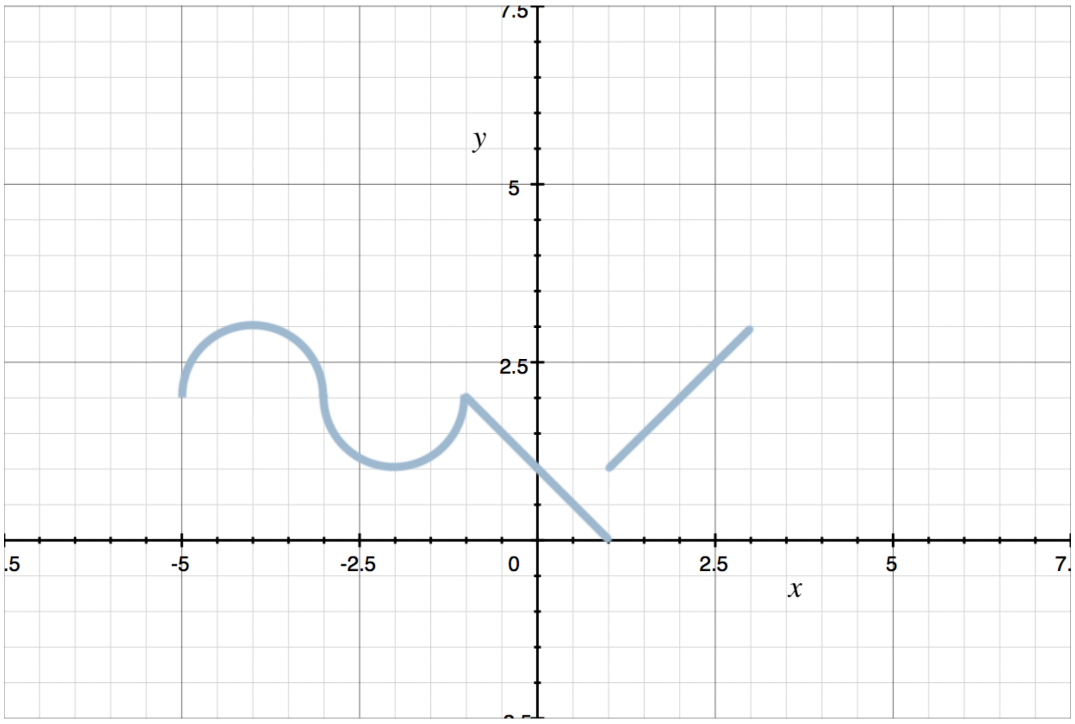
Solution: B

Even when the left- and right-hand limits are equal at a point, there could be a point discontinuity there, so choice A could be true.

For a function to be differentiable it must be continuous, but a function can be continuous without being differentiable (like at a cusp), so choice B cannot be true, but choice C can be.

Choice D is commonly true, functions are very often differentiable where they’re continuous.

■ 5. Given that the graph of f consists of semi-circles and line segments, which of the following statements is false?



A $f(x)$ is not differentiable when $x = -3$ because there is a vertical tangent line.

[✓ B] $f(x)$ is not differentiable when $x = -4$ and when $x = -2$ because there is a horizontal tangent line.

C $f(x)$ is not differentiable when $x = -1$ because there is a cusp.

D $f(x)$ is not differentiable when $x = 1$ because there is a discontinuity.

Solution: B

At $x = -3$ the function has a vertical tangent line and is therefore not differentiable, so answer choice A is true.

At $x = -4$ and $x = -2$, the function does have horizontal tangent lines. But a function is differentiable at horizontal tangent lines. The derivative is 0, but 0 isn't an undefined value. So answer choice B is false.

At $x = -1$ the function has a cusp and is therefore not differentiable, so answer choice C is true.

At $x = 1$ the function is discontinuous and is therefore not differentiable, so answer choice D is true.

■ 6. Find the derivative of $y = 3x^7 - 9x^2 + 21$.

A $y' = 21x^{-6} - 18x$

B $y' = 3x(7x^5 - 6x)$

C $y' = 21x^8 - 18x^2$

[✓ D] $y' = 21x^6 - 18x$

Solution: D

Apply power rule to differentiate the equation, one term at a time.

$$y' = 3(7)x^{7-1} - 9(2)x^{2-1} + 0$$

$$y' = 21x^6 - 18x^1$$

$$y' = 21x^6 - 18x$$

■ **7.** If $f(x)=\sqrt{x}+\dfrac{3}{\sqrt{x}}$, then find $f'(4)$.

$\boxed{\text{A}}$ $f'(4)=\dfrac{1}{16}$

B $f'(4)=\dfrac{5}{16}$

C $f'(4)=\dfrac{7}{2}$

D $f'(4)=\dfrac{7}{16}$

Solution: A

Rewrite the function using fractional exponents.

$$f(x)=x^{\frac{1}{2}}+3x^{-\frac{1}{2}}$$

Apply power rule.

$$f'(x)=\frac{1}{2}x^{-\frac{1}{2}}+3\left(-\frac{1}{2}\right)x^{-\frac{3}{2}}$$

$$f'(x)=\frac{1}{2x^{\frac{1}{2}}}-\frac{3}{2x^{\frac{3}{2}}}$$

$$f'(x)=\frac{1}{2\sqrt{x}}-\frac{3}{2(\sqrt{x})^3}$$

Evaluate the derivative at $x=4$ to find $f'(4)$.

$$f'(4)=\frac{1}{2\sqrt{4}}-\frac{3}{2(\sqrt{4})^3}$$

$$f'(4)=\frac{1}{2(2)}-\frac{3}{2(2)^3}$$

$$f'(4)=\frac{1}{4}-\frac{3}{16}$$

$$f'(4)=\frac{4}{16}-\frac{3}{16}$$

$$f'(4)=\frac{1}{16}$$

■ 8. Evaluate $\frac{d}{dx}3\cos x$.

A $3\sin x$

B $-\sin(3x)$

C $\sin(3x)$

[✓ D] $-3\sin x$

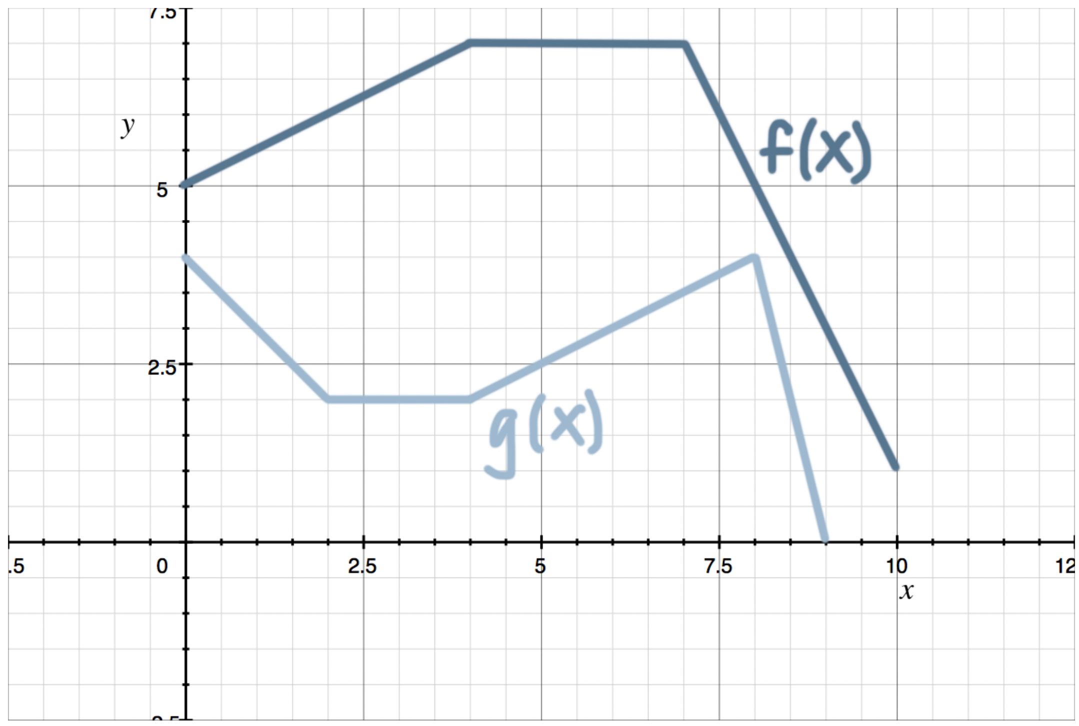
Solution: D

Apply the rule for the derivative of the cosine function.

$$\frac{d}{dx}(\cos u)=-\sin u$$

$$\frac{d}{dx}(3\cos x)=-3\sin x$$

■ 9. The graphs of f and g are shown below and consist only of line segments. Given that $h(x)=f(x)g(x)$, what is the value of $h'(1)$?



A $-\frac{1}{2}$

[✓ B] -4

C 7

D $\frac{33}{2}$

Solution: B

Apply the product rule to find $h'(x)$,

$$h'(x) = f'(x)g(x) + f(x)g'(x)$$

then evaluate at $x = 1$.

$$h'(1) = f'(1)g(1) + f(1)g'(1)$$

The values of $f(1)$ and $g(1)$ are given by the y-values on the graph, and $f'(1)$ and $g'(1)$ are given by the slopes at $x = 1$.

$$f(1)=5.5=\frac{11}{2} \qquad g(1)=3$$

$$f'(1)=\frac{1}{2} \qquad g'(1)=-1$$

Substituting into the derivative gives

$$h'(1)=\frac{1}{2}(3)+\frac{11}{2}(-1)$$

$$h'(1)=\frac{3}{2}-\frac{11}{2}$$

$$h'(1)=-\frac{8}{2}$$

$$h'(1)=-4$$

■ **10. Find the derivative of $y=\dfrac{x^2-x+1}{x^2+1}$.**

[✓ A] $y'=\dfrac{x^2-1}{(x^2+1)^2}$

B $y'=\dfrac{x-1}{(x^2+1)^2}$

C $y'=\dfrac{x^2-1}{x^2+1}$

D $y'=\dfrac{x^2}{(x^2+1)^2}$

Solution: A

Apply the quotient rule to find the derivative.

$$y' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$y' = \frac{(2x-1)(x^2+1) - (x^2-x+1)(2x)}{(x^2+1)^2}$$

Simplify the derivative.

$$y' = \frac{(2x^3+2x-x^2-1) - (2x^3-2x^2+2x)}{(x^2+1)^2}$$

$$y' = \frac{2x^3+2x-x^2-1-2x^3+2x^2-2x}{(x^2+1)^2}$$

$$y' = \frac{x^2-1}{(x^2+1)^2}$$

■ **11. Given $y = \tan x$, what is the equation of the tangent line at $x = \frac{\pi}{4}$?**

A $y = -\frac{\pi}{2}(x+1)$

B $y = x + 1$

$\boxed{\text{C}}$ $y = 2\left(x - \frac{\pi}{4}\right) + 1$

D $y = \frac{1}{2}\left(x - \frac{\pi}{4}\right) + 1$

Solution: C

Substitute $x = \frac{\pi}{4}$ into the function.

$$y = \tan\frac{\pi}{4} = 1$$

Find the derivative,

$$y' = \sec^2 x$$

then evaluate the derivative at $x = \frac{\pi}{4}$ to find the slope of the tangent line.

$$m = \sec^2\frac{\pi}{4} = \frac{1}{\cos^2\frac{\pi}{4}} = \frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2} = \frac{1}{\frac{2}{4}} = \frac{4}{2} = 2$$

The the tangent line equation is

$$y - 1 = 2\left(x - \frac{\pi}{4}\right)$$

$$y = 2\left(x - \frac{\pi}{4}\right) + 1$$

■ 12. Find the values of a and b that make the function differentiable.

$$f(x) = \begin{cases} 3ax^2 + 4 & x \le -1 \\ bx - a & x > -1 \end{cases}$$

Solution:

We have to find values of a and b that guarantee both continuity and smoothness at the break point, $x = -1$. First, find the derivative.

$$f'(x) = \begin{cases} 6ax & x \leq -1 \\ b & x > -1 \end{cases}$$

The function will be continuous when both pieces of the function are equivalent at $x = -1$.

$$3ax^2 + 4 = bx - a$$

$$3a(-1)^2 + 4 = b(-1) - a$$

$$3a + 4 = -b - a$$

$$4a + 4 = -b$$

$$b = -4a - 4$$

The function will be smooth when both pieces of the derivative are equivalent at $x = -1$.

$$6ax = b$$

$$6a(-1) = b$$

$$-6a = b$$

With two equations which are both equal to b, we can set them equal to each other to solve for a.

$$-4a-4=-6a$$

$$4a+4=6a$$

$$4=2a$$

$$a=2$$

Substitute $a=2$ into $-6a=b$.

$$-6(2)=b$$

$$b=-12$$

The values $a=2$ and $b=-12$ make the function continuous.

■ 13. Use the product rule to find the derivative of $h(x)=8x^3e^x$.

Solution:

Let $f(x)=8x^3$, $f'(x)=24x^2$, $g(x)=e^x$, and $g'(x)=e^x$. Then by the product rule, the derivative is

$$h'(x)=f'(x)g(x)+f(x)g'(x)$$

$$h'(x)=24x^2e^x+8x^3e^x$$

$$h'(x) = 8x^2e^x(3 + x)$$

■ 14. Let $f(x) = \dfrac{x^5 + 4x^3 - 11}{x^2}$. Find $f'(x)$.

Solution:

We could use quotient rule here to find the derivative, but it'll be easier to simplify the function first,

$$f(x) = \frac{x^5 + 4x^3 - 11}{x^2}$$

$$f(x) = \frac{x^5}{x^2} + \frac{4x^3}{x^2} - \frac{11}{x^2}$$

$$f(x) = x^3 + 4x - 11x^{-2}$$

and then instead apply power rule to find the derivative.

$$f'(x) = 3x^2 + 4 + 22x^{-3}$$

$$f'(x) = 3x^2 + 4 + \frac{22}{x^3}$$

Find a common denominator.

$$f'(x) = \frac{3x^5}{x^3} + \frac{4x^3}{x^3} + \frac{22}{x^3}$$

$$f'(x) = \frac{3x^5 + 4x^3 + 22}{x^3}$$