■ 1. Find the derivative of $y = \frac{4x}{(x^2-1)^3}$.

A $y' = -\frac{4x}{(x-1)^2}$

B $y' = -\frac{4x}{(x^2-1)^4}$

$\boxed{\text{C}}$ $y' = -\frac{4(5x^2+1)}{(x^2-1)^4}$

D $y' = -\frac{8x}{(x-1)^2}$

Solution: C

List out $f(x)$ and $g(x)$ and their derivatives.

$$f(x) = 4x$$

$$f'(x) = 4$$

and

$$g(x) = (x^2-1)^3$$

$$g'(x) = 6x(x^2-1)^2$$

Now we can plug these values directly into the quotient rule formula.

$$y' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$y' = \frac{(4)((x^2-1)^3) - (4x)(6x(x^2-1)^2)}{[(x^2-1)^3]^2}$$

$$y' = \frac{4(x^2-1)^3 - 24x^2(x^2-1)^2}{(x^2-1)^6}$$

Within the fraction, we have a common factor of $(x^2-1)^2$, so cancel that out.

$$y' = \frac{4(x^2-1)-24x^2}{(x^2-1)^4}$$

$$y' = \frac{4x^2-4-24x^2}{(x^2-1)^4}$$

$$y' = \frac{-4-20x^2}{(x^2-1)^4}$$

$$y' = -\frac{4(5x^2+1)}{(x^2-1)^4}$$

■ 2. Given the function $g(x) = (2x+h(3x))^2$, which of the following represents $g'(x)$?

A $2(2x+h(3x)) \cdot 3h'(x)$

B $2(2+3h'(3x))$

[✓ C] $2(2x+h(3x))(2+3h'(3x))$

D $2(2x+h(3x))(h'(x))$

Solution: C

Apply the power rule for u^n,

$$\frac{d}{dx}(u^n) = nu^{n-1}u'$$

with $u = (2x+h(3x))$. Then the derivative is

$$2(2x+h(3x))^{2-1}\frac{d}{dx}(2x+h(3x))$$

$$2(2x+h(3x))\left(2+h'(3x)\frac{d}{dx}(3x)\right)$$

$$2(2x+h(3x))\left(2+h'(3x)(3)\right)$$

$$2(2x+h(3x))(2+3h'(3x))$$

■ 3. Evaluate $\frac{d}{dx}[\sin^4(2x)]$.

A $8\sin^3(2x)$

B $4\cos^3(2x)$

C $4\sin^3(2x)\cos(2x)$

D $8\sin^3(2x)\cos(2x)$

Solution: D

Apply the power rule for u^n,

$$\frac{d}{dx}(u^n)=nu^{n-1}u'$$

with $u=\sin(2x)$. Then the derivative is

$$4\sin^3(2x)\cdot\frac{d}{dx}[\sin(2x)]$$

Apply the chain rule to find the derivative of $\sin(2x)$.

$$4\sin^3(2x)\cos(2x)(2)$$

$$8\sin^3(2x)\cos(2x)$$

■ 4. Find the derivative of the exponential function $y = e^{\sqrt{x+1}}$.

[✓ A] $y' = \dfrac{e^{\sqrt{x+1}}}{2\sqrt{x+1}}$

B $y' = \dfrac{e^{\sqrt{x}}}{2\sqrt{x+1}}$

C $y' = \dfrac{e^{\sqrt{x+1}}}{\sqrt{x+1}}$

D $y' = e^{\sqrt{x+1}}$

Solution: A

Make a substitution, letting $u = \sqrt{x+1}$ and

$$u' = \frac{1}{2\sqrt{x+1}}$$

Then the function is

$$y = e^u$$

and the derivative is

$$y' = e^u \cdot u'$$

$$y' = e^{\sqrt{x+1}} \cdot \frac{1}{2\sqrt{x+1}}$$

$$y' = \frac{e^{\sqrt{x+1}}}{2\sqrt{x+1}}$$

■ 5. Find the derivative of the logarithmic function $y = \ln(x^2 - 5x)$.

A $\quad y' = \dfrac{2x+5}{x^2+5x}$ B $\quad y' = \dfrac{2x-5}{x^2-5}$

C $\quad y' = \dfrac{5-2x}{x^2-5x}$ $\boxed{\text{D}}$ $\quad y' = \dfrac{2x-5}{x^2-5x}$

Solution: D

Let $u = x^2 - 5x$ and $u' = 2x - 5$. Then the function is

$$y = \ln u$$

and the derivative is

$$y' = \frac{1}{u} \cdot u'$$

$$y' = \frac{1}{x^2 - 5x} \cdot (2x - 5)$$

$$y' = \frac{2x - 5}{x^2 - 5x}$$

■ 6. Find the slope of the tangent line to the curve $x^2 + xy^2 + y = 25$ at the point $(2,3)$.

$\boxed{\text{A}}$ $\quad -1$ B $\quad 38$

C $\quad -4$ D $\quad 11$

Solution: A

Use implicit differentiation. Whenever we differentiate y, we'll multiply by y', and when we differentiate y', we'll multiply by y''. Pay particular attention to applying the product rule to differentiate xy^2. Also note that the derivative of 25 is 0, as it's a common mistake to have the 25 make it through the differentiation.

$$\frac{d}{dx}[x^2+xy^2+y=25]$$

$$2x+[(1)(y^2)+(x)(2y)(y')]+y'=0$$

$$2x+y^2+2xyy'+y'=0$$

It's generally a good idea to plug numbers in as soon as possible in order to simplify the function, so we'll substitute $(x,y)=(2,3)$.

$$2(2)+3^2+2(2)(3)y'+y'=0$$

$$4+9+12y'+y'=0$$

$$13+13y'=0$$

Now we can solve algebraically.

$$13y'=-13$$

$$y'=-1$$

■ 7. Given that $5x+\cot y=3$, find $\dfrac{dy}{dx}$.

A $5\cos^2 y$

[✓ B] $5\sin^2 y$

C $5\csc y\cot y$

D $\dfrac{5}{1+25y^2}$

Solution: B

Use implicit differentiation, noting that the derivative of 3 is 0, and the derivative of $\cot y$ is $-\csc^2 y$.

$$\frac{d}{dx}[5x + \cot y = 3]$$

$$5 - \csc^2 y \cdot y' = 0$$

Solve for y'.

$$-\csc^2 y \cdot y' = -5$$

$$y' = \frac{5}{\csc^2 y}$$

Remember that

$$\sin y = \frac{1}{\csc y}$$

Then

$$y' = 5\sin^2 y$$

■ 8. Given that f and g are inverses of each other, find $g'(9)$ given $f(-3) = 9$, $f'(-3) = 4$, $f(9) = -3$, and $f'(9) = 6$.

A $\frac{1}{6}$

B $\frac{1}{9}$

[✓ C] $\frac{1}{4}$

D $-\frac{1}{6}$

Solution: C

Remember that if f and g are inverses of each other, then $f(g(x)) = x$. If we differentiate that relationship using chain rule, we get

$$\frac{d}{dx}[f(g(x)) = x]$$

$$f'(g(x))g'(x) = 1$$

$$g'(x) = \frac{1}{f'(g(x))}$$

We want to find $g'(9)$, so we'll substitute $x = 9$.

$$g'(9) = \frac{1}{f'(g(9))}$$

Because $f(-3) = 9$, and f and g are inverses of each other, we know $g(9) = -3$.

$$g'(9) = \frac{1}{f'(-3)}$$

$$g'(9) = \frac{1}{4}$$

■ 9. Find the derivative of the inverse trig function $f(x) = \tan^{-1}(x^2 - 1)$.

$\boxed{\text{A}}$ $f'(x) = \dfrac{2x}{x^4 - 2x^2 + 2}$

B $f'(x) = \dfrac{1}{x^4 - 2x^2 + 2}$

C $f'(x) = \dfrac{2x}{1 + 4x^2}$

D $f'(x) = \dfrac{1}{1 + 4x^2}$

Solution: A

Apply the formula for the derivative of inverse tangent, with $g(x) = x^2 - 1$, in order to differentiate the function.

$$y' = \frac{g'(x)}{1 + [g(x)]^2}$$

$$y' = \frac{2x}{1 + (x^2 - 1)^2}$$

Simplify the derivative.

$$y' = \frac{2x}{1 + (x^4 - 2x^2 + 1)}$$

$$y' = \frac{2x}{x^4 - 2x^2 + 2}$$

■ 10. Find the derivative of the logarithmic function $y = \ln \sqrt[3]{2x^3 - 5}$.

A $y' = \frac{2x}{2x^3 - 5}$

B $y' = \frac{x^2}{2x^3 - 5}$

C $y' = \frac{2x^2}{2x^3 + 5}$

$\boxed{\text{D}}$ $y' = \frac{2x^2}{2x^3 - 5}$

Solution: D

Let $u = \sqrt[3]{2x^3 - 5}$, so $u = (2x^3 - 5)^{\frac{1}{3}}$. Then u' is

$$u' = \frac{1}{3}(2x^3 - 5)^{-\frac{2}{3}}(6x^2)$$

$$u' = 2x^2(2x^3 - 5)^{-\frac{2}{3}}$$

The function is $y = \ln u$, and its derivative is

$$y' = \frac{1}{u} \cdot u'$$

$$y' = \frac{1}{(2x^3 - 5)^{\frac{1}{3}}} \cdot 2x^2(2x^3 - 5)^{-\frac{2}{3}}$$

$$y' = \frac{2x^2}{(2x^3 - 5)^{\frac{1}{3}}(2x^3 - 5)^{\frac{2}{3}}}$$

$$y' = \frac{2x^2}{(2x^3 - 5)^1}$$

$$y' = \frac{2x^2}{2x^3 - 5}$$

■ 11. Given $\ln y + x^2y^2 = 9$, determine an equation for $\frac{dy}{dx}$ in terms of x and y. Then find an equation for all lines tangent to the curve when $y = 1$.

Solution:

Use implicit differentiation.

$$\frac{d}{dt}[\ln y + x^2y^2 = 9]$$

$$\frac{1}{y}y' + 2xy^2 + 2x^2yy' = 0$$

Solve for y'.

$$\frac{1}{y}y' + 2x^2yy' = -2xy^2$$

$$\left(\frac{1}{y} + 2x^2y\right)y' = -2xy^2$$

$$y' = -\frac{2xy^2}{\frac{1}{y} + 2x^2y}$$

Simplify by clearing the fraction in the denominator.

$$y' = -\frac{2xy^2}{\frac{1}{y} + 2x^2y}\left(\frac{y}{y}\right)$$

$$y' = -\frac{2xy^3}{1 + 2x^2y^2}$$

To find any lines tangent to the curve at $y = 1$, substitute $y = 1$ into the original equation.

$$\ln 1 + x^2(1)^2 = 9$$

$$x^2 = 9$$

$$x = \pm 3$$

Two tangent lines exist, one at $(-3,1)$ and the other at $(3,1)$. To find the slope of each tangent line, substitute the point of tangency into the derivative equation. At $(-3,1)$ the slope is

$$y' = -\frac{2(-3)(1)^3}{1+2(-3)^2(1)^2}$$

$$y' = \frac{6}{19}$$

and at $(3,1)$ the slope is

$$y' = -\frac{2(3)(1)^3}{1+2(3)^2(1)^2}$$

$$y' = -\frac{6}{19}$$

Then the equations of the two tangent lines are

$$y - 1 = \frac{6}{19}(x+3)$$

$$y - 1 = -\frac{6}{19}(x-3)$$

■ 12. Given that $f(x) = e^x \sin x$,

a. Find average rate of change on the interval $\left[-\pi, \frac{\pi}{2}\right]$.

b. Determine the equation of the tangent line to the curve at $x = \pi$.

c. Determine all values on the interval $[0,2\pi]$ where the second derivative is 0.

Solution:

a. Use the difference quotient to find average rate of change.

$$\frac{f\left(\frac{\pi}{2}\right)-f(-\pi)}{\frac{\pi}{2}-(-\pi)}$$

$$\frac{e^{\frac{\pi}{2}}\sin\left(\frac{\pi}{2}\right)-e^{-\pi}\sin(-\pi)}{\frac{\pi}{2}+\pi}$$

$$\frac{e^{\frac{\pi}{2}}(1)-e^{-\pi}(0)}{\frac{\pi}{2}+\frac{2\pi}{2}}$$

$$\frac{e^{\frac{\pi}{2}}}{\frac{3\pi}{2}}$$

$$\frac{2e^{\frac{\pi}{2}}}{3\pi}$$

b. To find the equation of the tangent line at $x=\pi$, first substitute $x=\pi$ into the original equation.

$$f(\pi)=e^{\pi}\sin\pi$$

$$f(\pi)=e^{\pi}(0)$$

$$f(\pi)=0$$

Then find the derivative,

$$f'(x)=(e^{x})(\sin x)+(e^{x})(\cos x)$$

$$f'(x) = e^x \sin x + e^x \cos x$$

and evaluate it at $x = \pi$ to find the slope there.

$$f'(\pi) = e^\pi \sin \pi + e^\pi \cos \pi$$

$$f'(\pi) = e^\pi(0) + e^\pi(-1)$$

$$f'(\pi) = -e^\pi$$

Then build the tangent line equation.

$$y - 0 = -e^\pi(x - \pi)$$

$$y = -e^\pi(x - \pi)$$

c. The first derivative is $f'(x) = e^x \sin x + e^x \cos x$, so the second derivative is

$$f''(x) = [(e^x)(\sin x) + (e^x)(\cos x)] + [(e^x)(\cos x) + (e^x)(-\sin x)]$$

$$f''(x) = (e^x \sin x + e^x \cos x) + (e^x \cos x - e^x \sin x)$$

$$f''(x) = e^x \cos x + e^x \cos x$$

$$f''(x) = 2e^x \cos x$$

Because e^x will never equal 0, the second derivative is equal to 0 when $\cos x = 0$, which happens in the interval $[0,2\pi]$ at

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$