■ 1. Given $f(x) = 9 - x^2$ on the interval $[-3,0]$, which of the following would yield an area approximation that's an underestimate.

I. Trapezoidal rule

II. Left endpoint rectangle

III. Right endpoint rectangle

A	I only	B	III only
C	I and II	D	I and III

Solution: C

Find the derivative of $f(x) = 9 - x^2$, then solve for critical points.

$$f'(x) = -2x$$

$$-2x = 0$$

$$x = 0$$

Choose a value to the left of $x = 0$ in the interval $[-3,0]$, and test it in the first derivative.

$$f'(-1) = -2(-1) = 2$$

Because we get a positive value, $f(x)$ is increasing over the entire interval. For increasing functions, left endpoints give an underestimate, while right endpoints give an overestimate.

To determine whether the trapezoidal rule will give an over or underestimate, we need to consider the concavity of the function, so we find the second derivative.

$$f''(x) = -2$$

Because the second derivative is constant and negative, the function is concave down everywhere. For functions which are concave down, the trapezoidal rule gives an underestimate.

■ 2. Consider $f(x) = \int_{-2}^{x^2} \sqrt{1+t^2}\ dt$. Find $f'(x)$.

A $\sqrt{1+x^4}$

B $\sqrt{1+x^4} - \sqrt{5}$

C $2x\sqrt{1+x^4}$

D $2x(\sqrt{1+x^4} - \sqrt{5})$

Solution: C

The Fundamental Theorem of Calculus tells us that, when we take the derivative of an integral, we can say

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}\left(\int_{-2}^{x^2} \sqrt{1+t^2}\ dt\right)$$

$$f'(x) = \sqrt{1+(x^2)^2}\left(\frac{d}{dx}(x^2)\right)$$

$$f'(x) = \sqrt{1+x^4}(2x)$$

$$f'(x) = 2x\sqrt{1+x^4}$$

■ 3. Given $\int_0^5 f(x)\ dx = 9$ and $\int_2^5 f(x)\ dx = -1$, which statement is true?

A $\int_2^0 f(x)\ dx = -10$

B $\int_0^2 f(x)\ dx = -10$

C $\int_0^2 f(x)\ dx = 8$

D $\int_0^2 f(x)\ dx = -8$

Solution: A

If total area under the curve on $[0,5]$ is 9, and the part of that area that lies on $[2,5]$ is -1, then the area on $[0,2]$ must be 10.

$$\int_0^5 f(x)\ dx = \int_0^2 f(x)\ dx + \int_2^5 f(x)\ dx$$

$$9 = \int_0^2 f(x)\ dx + (-1)$$

$$9 = \int_0^2 f(x)\ dx - 1$$

$$10 = \int_0^2 f(x)\ dx$$

This doesn't match any of the answer choices, but if we reverse the limits of integration, then we have to multiply by -1.

$$10 = -\int_2^0 f(x)\ dx$$

$$-10 = \int_2^0 f(x)\ dx$$

$$\int_2^0 f(x)\ dx = -10$$

■ 4. Evaluate the following: $\int \frac{x}{\sqrt{x-4}}\ dx$

A $\frac{2}{3}(x-4)^{\frac{3}{2}} - 2(x-4)^{\frac{1}{2}} + C$

B $\left(\frac{3}{2}x-4\right)^{\frac{3}{2}} + 2(x-4)^{\frac{1}{2}} + C$

C $\frac{x^2\sqrt{x-4}}{2} + C$

D $\frac{2}{3}(x-4)^{\frac{3}{2}} + 8(x-4)^{\frac{1}{2}} + C$

Solution: D

Use a substitution, with $u = x - 4$, and therefore $du = dx$ and $x = u + 4$. Substituting these values into the integral gives

$$\int \frac{u+4}{\sqrt{u}}\ du$$

Split the fraction, rewrite each one as a power function, then integrate.

$$\int \frac{u}{\sqrt{u}} + \frac{4}{\sqrt{u}}\ du$$

$$\int u^{\frac{1}{2}} + 4u^{-\frac{1}{2}}\ du$$

$$\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + \frac{4u^{\frac{1}{2}}}{\frac{1}{2}} + C$$

Clear the fractions, then back-substitute.

$$\frac{2}{3}u^{\frac{3}{2}} + 8u^{\frac{1}{2}} + C$$

$$\frac{2}{3}(x-4)^{\frac{3}{2}} + 8(x-4)^{\frac{1}{2}} + C$$

■ 5. Which of the following is an antiderivative of $y = \sin x \cos x$

A $\frac{1}{4}\sin^2(2x) + 8$

B $\frac{1}{2}\cos^2 x + 5$

C $\frac{1}{2}\sin^2 x + 6$

D $-\frac{1}{2}\sin^2 x + 4$

Solution: C

Differentiate each answer choice to see which gives the correct answer. You should be able to rule out answer choice A, because of the $2x$ argument. But differentiating each answer choice gives

A. $\frac{d}{dx}\left[\frac{1}{4}\sin^2(2x) + 8\right]$

$$= \frac{1}{4}(2\sin(2x))\frac{d}{dx}(\sin(2x)) = \frac{1}{2}\sin(2x)\cos(2x)(2) = \sin(2x)\cos(2x)$$

B. $\frac{d}{dx}\left[\frac{1}{2}\cos^2 x + 5\right]$

$$= \frac{1}{2}(2\cos x)\frac{d}{dx}(\cos x) = \cos x(-\sin x) = -\sin x\cos x$$

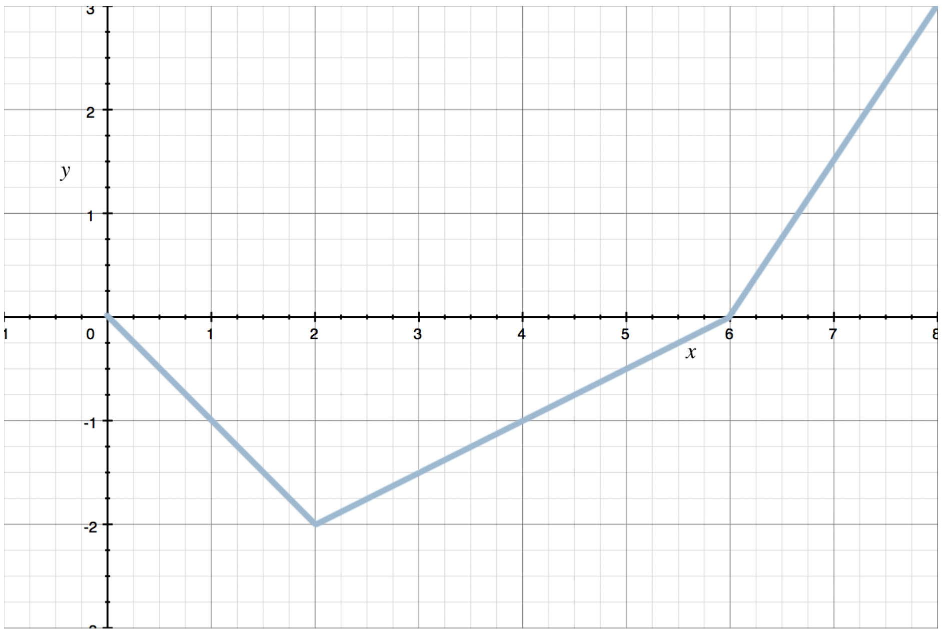
C. $\frac{d}{dx}\left[\frac{1}{2}\sin^2 x + 6\right]$

$$= \frac{1}{2}(2\sin x)\frac{d}{dx}(\sin x) = \sin x\cos x$$

D. $\frac{d}{dx}\left[-\frac{1}{2}\sin^2 x + 4\right]$

$$= -\frac{1}{2}(2\sin x)\frac{d}{dx}(\sin x) = -\sin x\cos x$$

■ 6. Given the graph of f and $g(x) = \int_0^x f(t)\ dt$, what is the value of $g''(4)$?



A 0

B $\frac{1}{2}$

C -5

D -1

Solution: B

From the Fundamental Theorem of Calculus, and the relationship we've been given,

$$g(x) = \int_0^x f(t)\ dt$$

we know $g'(x) = f(x)$, so $g''(4) = f'(4)$. The line segment that defines the graph through $x = 4$ has slope $m = \frac{1}{2}$ (because that line segment passes through $(2, -2)$ and $(6,0)$).

■ 7. $R(t) = 5 - \sin x$ represents the rate at which sucrose is consumed by a colony of bacteria, measured in mg/day. How much sucrose did the bacteria consume over the first 6 days?

A 29.995 mg

B 1.123 mg

C 5.123 mg

D 10.127 mg

Solution: A

To find the total amount of sucrose consumed, we can integrate the function defining the rate of consumption over the given interval.

$$\int_0^6 5-\sin x\ dx$$

Integrate, then evaluate over the interval.

$$5x+\cos x\Big|_0^6$$

$$5(6)+\cos 6-(5(0)+\cos 0)$$

$$30+\cos 6-\cos 0$$

29.995 mg/day

■ 8. If $f(5)=7$ and $f'(x)=x^{0.3}+e^{0.2x}$, what is the value of $f(1)$?

A 2.221

B 12.948

C 19.984

D -5.948

Solution: D

Use Part 2 of the Fundamental Theorem of Calculus.

$$\int_a^b f'(x)=f(b)-f(a)$$

$$\int_1^5 x^{0.3}+e^{0.2x}\ dx=f(5)-f(1)$$

Evaluating the integral,

$$\frac{1}{1.3}x^{1.3}+\frac{1}{0.2}e^{0.2x}\Big|_1^5=f(5)-f(1)$$

$$\frac{1}{1.3}(5)^{1.3}+\frac{1}{0.2}e^{0.2(5)}-\left(\frac{1}{1.3}(1)^{1.3}+\frac{1}{0.2}e^{0.2(1)}\right)=f(5)-f(1)$$

$$\frac{1}{1.3}(5)^{1.3}+5e-\frac{1}{1.3}-5e^{0.2}=f(5)-f(1)$$

and substituting $f(5)=7$ gives

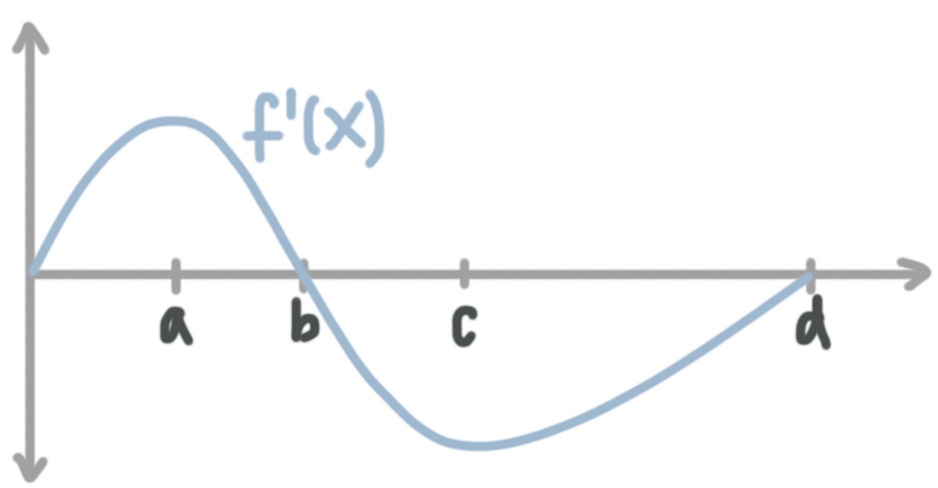
$$12.948=7-f(1)$$

$$12.948-7=-f(1)$$

$$f(1)=7-12.948$$

$$f(1)=-5.948$$

■ 9. Given the graph of $f'(x)$ below, on what interval, if any, is $f(x)$ both increasing and concave down, if $f'(x)$ has horizontal tangents at $x=a$ and $x=c$?



A (a,b)

B $(0,a)$

C (c,d)

D There is no interval where $f(x)$ is increasing and concave down

Solution: A

For $f(x)$ to be increasing, we need $f'(x)$ to be positive, which only occurs on the interval $(0,b)$. For $f(x)$ to be concave down, we need $f'(x)$ to be decreasing, which only occurs on the interval (a,c).

The intersection of these intervals is (a,b).

■ 10. Evaluate $\int_{-1}^{2} x(x^2-1)^3\ dx$

A $\frac{81}{4}$

B $\frac{81}{8}$

C 3

D $\frac{15}{4}$

Solution: B

Use a substitution with $u=x^2-1$, $du=2x\ dx$, and $dx=\frac{du}{2x}$. First find the bounds for u.

$$u(-1)=(-1)^2-1=1-1=0$$

$$u(2)=2^2-1=4-1=3$$

Substitute into the integral.

$$\int_0^3 xu^3 \frac{du}{2x}$$

$$\frac{1}{2}\int_0^3 u^3\ du$$

Integrate, then evaluate over the interval.

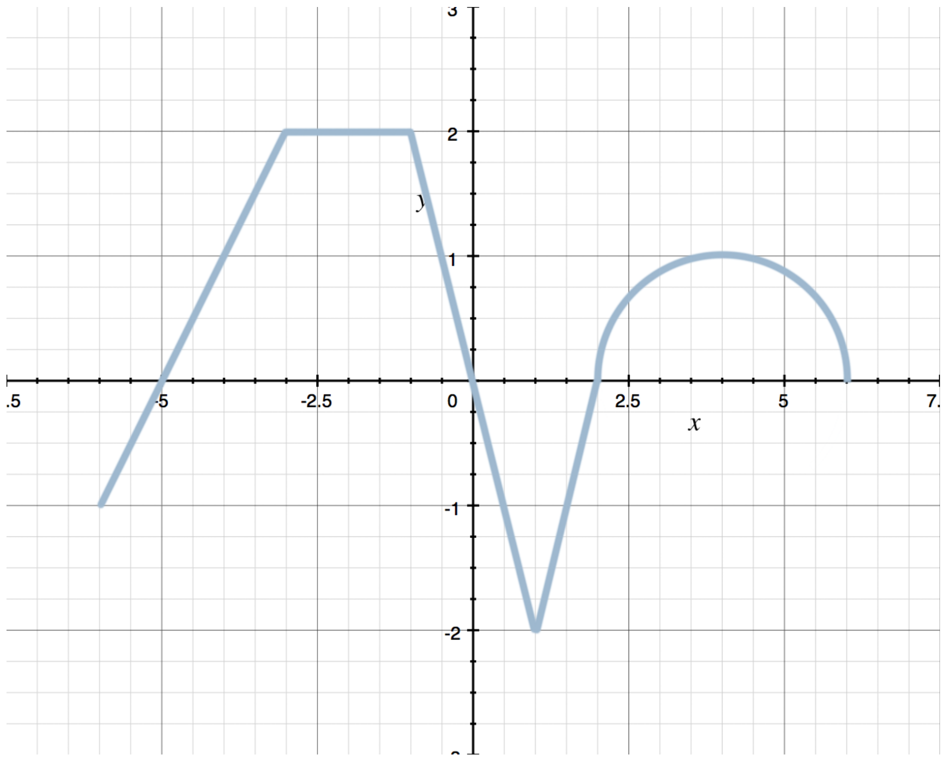
$$\frac{1}{2}\left(\frac{1}{4}u^4\right)\Bigg|_0^3$$

$$\frac{1}{8}u^4\Bigg|_0^3$$

$$\frac{1}{8}(3^4) - \frac{1}{8}(0^4)$$

$$\frac{81}{8}$$

■ 11. The graph of f, shown below, consists exclusively of line segments and a semi-circle. The function g is defined by $g(x) = \int_{-2}^{x} f(t)\ dt$. Answer each question and justify your answers.



a) Find any relative extrema of $g(x)$ on the interval $(-6,6)$.

b) Does $g(x)$ have an inflection point at $x = 1$, 2, or 4?

c) Evaluate $g'(-4)$ and $g''(-4)$.

d) Determine the absolute maximum of g on the interval $-6 \leq x \leq 6$.

Solution:

The equation $g(x) = \int_{-2}^{x} f(t)\ dt$ implies that $g'(x) = f(x)$ and therefore that $g''(x) = f'(x)$, so the graph shows $g'(x)$.

a) Relative extrema of $g(x)$ can occur where $g'(x)$ crosses the x-axis, so $x = -5, 0, 2$. At $x = -5$ and $x = 2$, the graph of $g'(x)$ switches from negative to positive, which means $g(x)$ has relative minima at those points. Because the graph of $g'(x)$ switches from positive to negative at $x = 0$, $g(x)$ would have a relative maximum there.

b) Wherever the graph of $g'(x)$ (which is $f(x)$) switches from increasing to decreasing, or vice versa, $g(x)$ will have an inflection point. Therefore, $g(x)$ will have an inflection point at $x = 1$, where $f(x)$ switches from decreasing to increasing, and also at $x = 4$, where $f(x)$ switches from increasing to decreasing. There is no inflection point at $x = 2$ because the graph doesn't change direction there.

c) Because $g'(x) = f(x)$, we can see directly from the graph that $g'(-4) = f(-4) = 1$. We also know $g''(-4) = f'(-4)$, but the second derivative will be given by the slope of the graph. At $x = -4$, the slope is 1, so $g''(-4) = 1$.

d) The absolute maximum can occur at the endpoints or at a relative maximum, so we will consider $g(-6)$, $g(6)$, and $g(0)$. Using the integral we were given, $g(-6)$ is

$$g(-6) = \int_{-2}^{-6} f(t)\ dt$$

$$g(-6) = -\int_{-6}^{-2} f(t)\ dt$$

$$g(-6) = -\left(\int_{-6}^{-5} f(t)\ dt + \int_{-5}^{-3} f(t)\ dt + \int_{-3}^{-2} f(t)\ dt\right)$$

$$g(-6) = -\left(-\frac{1}{2} + 2 + 2\right)$$

$$g(-6) = -\frac{7}{2}$$

$g(6)$ is

$$g(6) = \int_{-2}^{6} f(t)\ dt$$

$$g(6)=\int_{-2}^{-1} f(t)\ dt+\int_{-1}^{1} f(t)\ dt+\int_{1}^{2} f(t)\ dt+\int_{2}^{6} f(t)\ dt$$

$$g(6)=2+0+(-1)+\frac{1}{2}\pi(2)^2$$

$$g(6)=1+2\pi$$

And $g(0)$ is

$$g(0)=\int_{-2}^{0} f(t)\ dt$$

$$g(0)=\int_{-2}^{-1} f(t)\ dt+\int_{-1}^{0} f(t)\ dt$$

$$g(0)=2+1$$

$$g(0)=3$$

Of these, $g(6)$ is the largest, so the absolute maximum on the interval is $1+2\pi$.

■ 12. The temperature of a cake removed from an oven is modeled by a strictly decreasing function $C(t)$ that is twice differentiable, where t is measured in minutes and $C(t)$ in degrees F. Select values of $C(t)$ are given in the table below. Answer each of the following questions, interpreting the meaning of each answer in the context of the problem.

t	0	2	5	9	12
C(t)	350	310	265	217	187

a) Using the data in the table, approximate $C'(7)$.

b) Using a Riemann sum with right endpoints, approximate the value of $\frac{1}{12}\int_0^{12} C(t)\ dt$.

c) Using the data in the table, evaluate $\int_0^5 C'(t)\ dt$.

Solution:

a) $C'(7) \approx \frac{C(9)-C(5)}{9-5} \approx \frac{217-265}{4} \approx -12$. The temperature of the cake is decreasing at approximately -12° F per minute when $t=7$ minutes.

b) Using right endpoints, we'll use every value of $C(t)$ from the table other than 350. We have to consider the width of each subinterval individually. $\frac{1}{12}\int_0^{12} C(t)\ dt \approx \frac{1}{12}[2(310)+3(265)+4(217)+3(187)] \approx 237$. This means that the average temperature of the cake over the first 12 minutes was approximately 237° F.

c) $\int_0^5 C'(t)\ dt = C(5)-C(0) = 265-350 = -85$. This means that the cake has cooled down 85° F between $t=0$ and $t=5$.