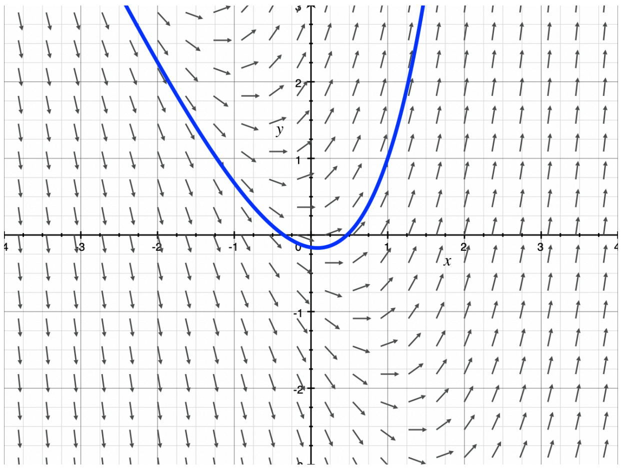
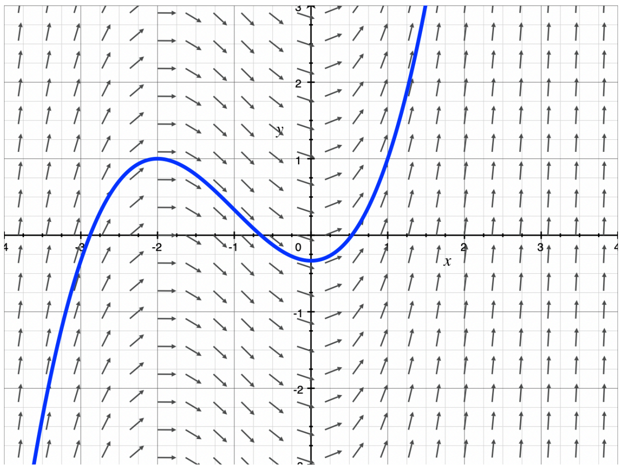
■ 1. Sketch the direction field for $y' = y - 2x$ and the solution curve at $(1,1)$.

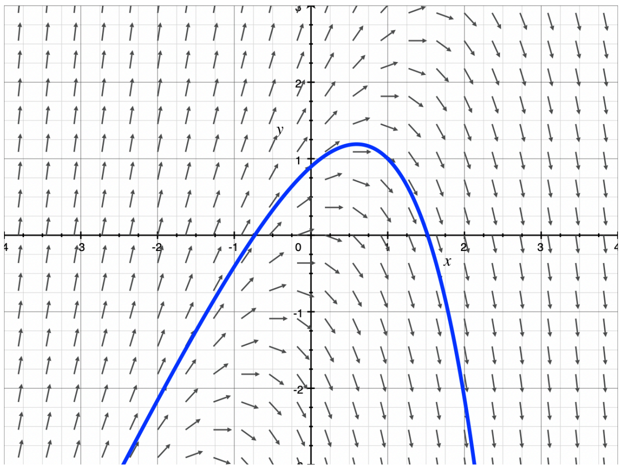
A



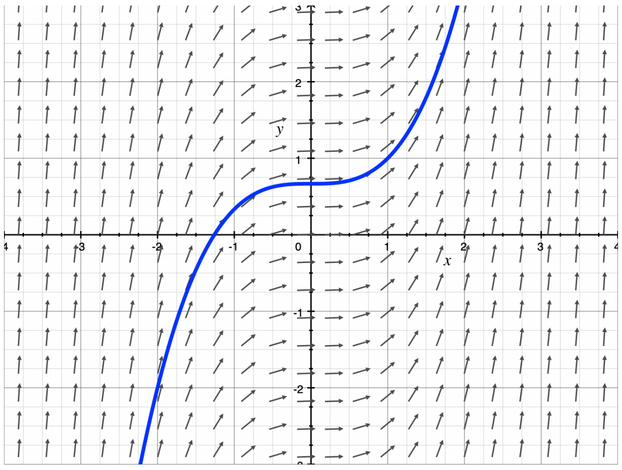
B



C



D



Solution: C

We'll build several tables, keeping x constant in each table, starting with a table for $x = -2$, exploring y-values on the interval $[-2,2]$, and then pairing those x and y values together to solve for values of y'.

x	-2	-2	-2	-2	-2
y	-2	-1	0	1	2
y'	2	3	4	5	6

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x	-1	-1	-1	-1	-1
y	-2	-1	0	1	2
y'	0	1	2	3	4

x	0	0	0	0	0
y	-2	-1	0	1	2
y'	-2	-1	0	1	2

x	1	1	1	1	1
y	-2	-1	0	1	2
y'	-4	-3	-2	-1	0

x	2	2	2	2	2
y	-2	-1	0	1	2
y'	-6	-5	-4	-3	-2

The values of y' that we found represent the slope of the function at the corresponding point (x, y). For example, in this last table, the slope of the function at $(2, -2)$ is -6, so we'd draw a small, short line with slope -6 right at $(2, -2)$. Plotting all of the other point-slope pairs, the direction field starts to look something like this:

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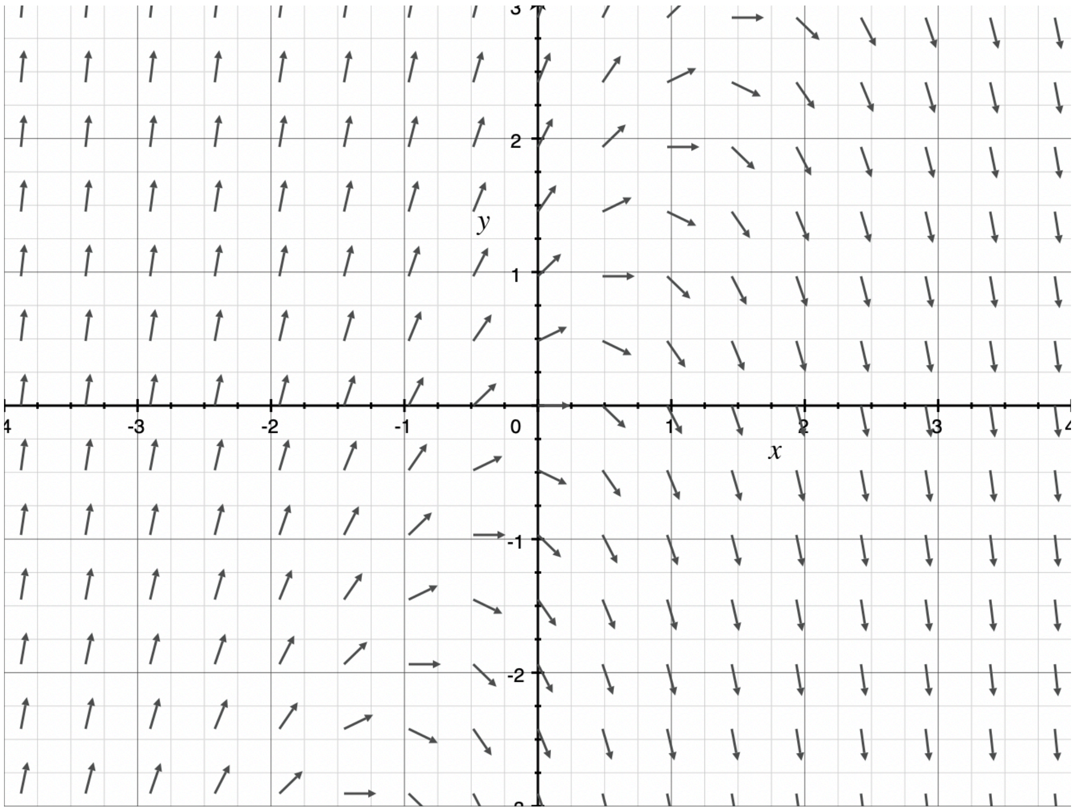
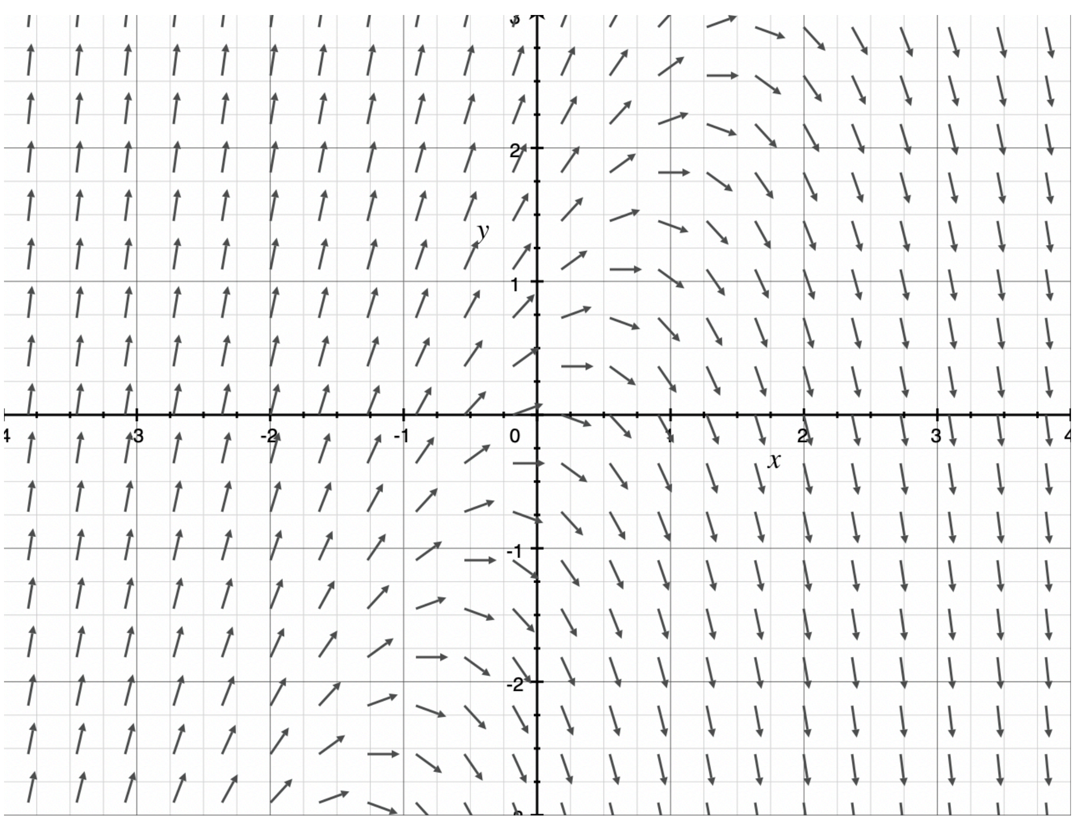


If we add more points, maybe ones that are half-way between those that we already found, a more complete direction field should look something like this:



Sketching the solution curve through $(1,1)$ gives

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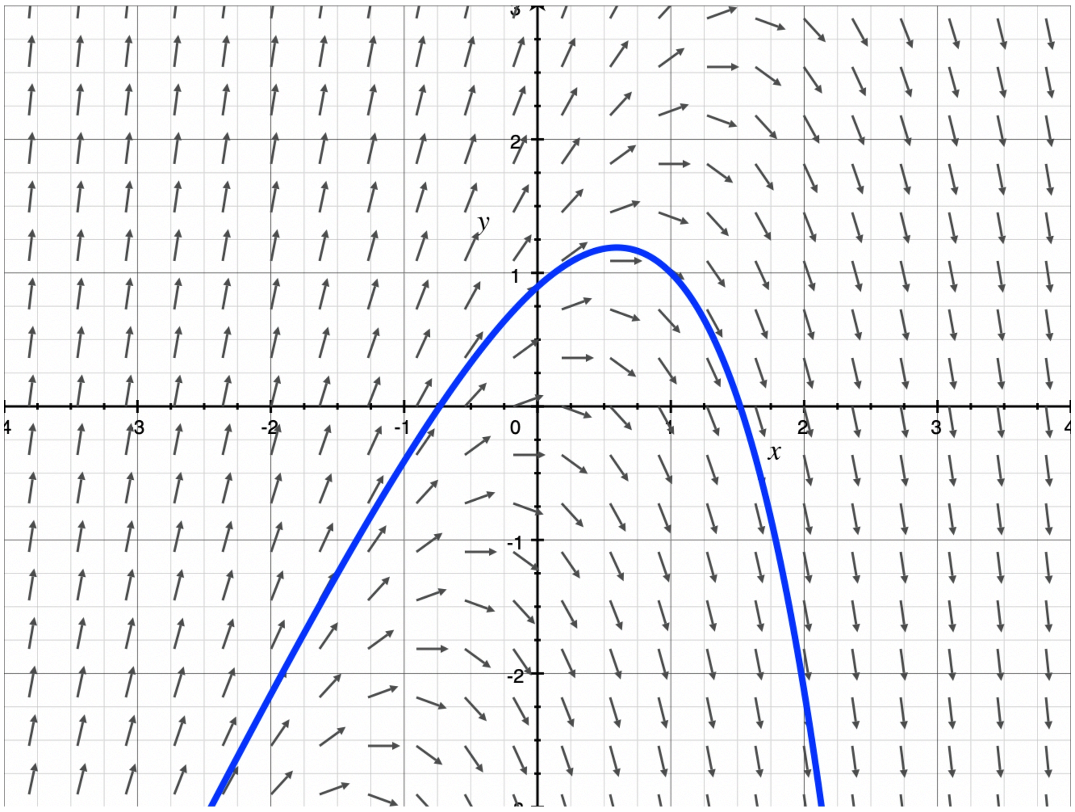
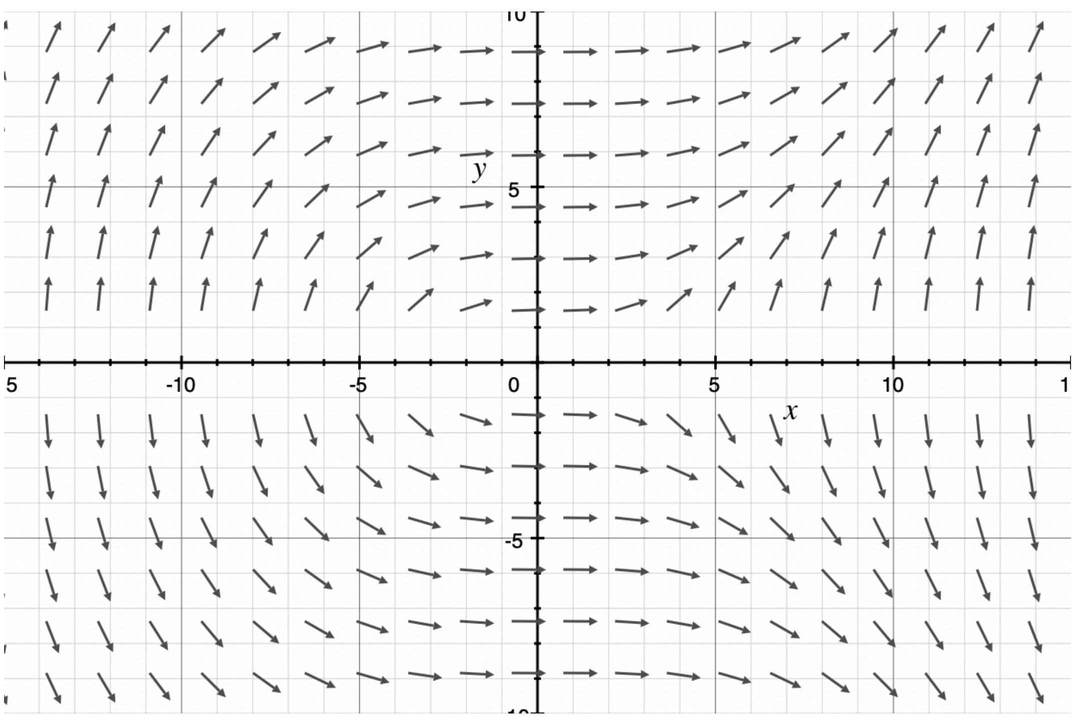


■ 2. Shown below is the slope field for which of the following differential equations?



A $\frac{dy}{dx} = \frac{-x^2}{5y}$

B $\frac{dy}{dx} = \frac{x^2}{10y}$

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C $\dfrac{dy}{dx}=\dfrac{x^3}{10y}$

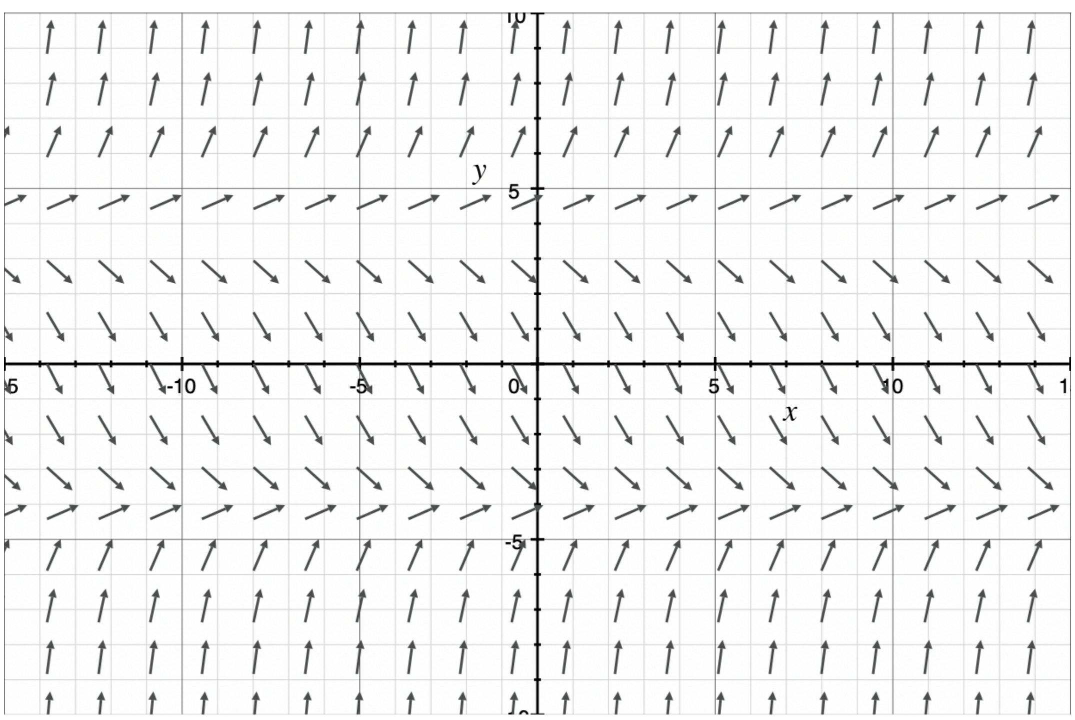
D $\dfrac{dy}{dx}=\dfrac{5}{y}$

Solution: B

Along the y-axis, we have horizontal line segments, which means the differential must equal 0 when $y=0$, which eliminates answer choice D.

In the first quadrant, the field has only positive slopes, which eliminates answer choice A. All slopes are positive in the second quadrant as well, which eliminates answer choice C.

■ 3. Find the limit $\lim_{x\to-\infty} f(x)$ from the slope field shown below for $\dfrac{dy}{dx}=\dfrac{y^2-16}{8}$ that satisfies the initial condition $f(-2)=2$.



A -4

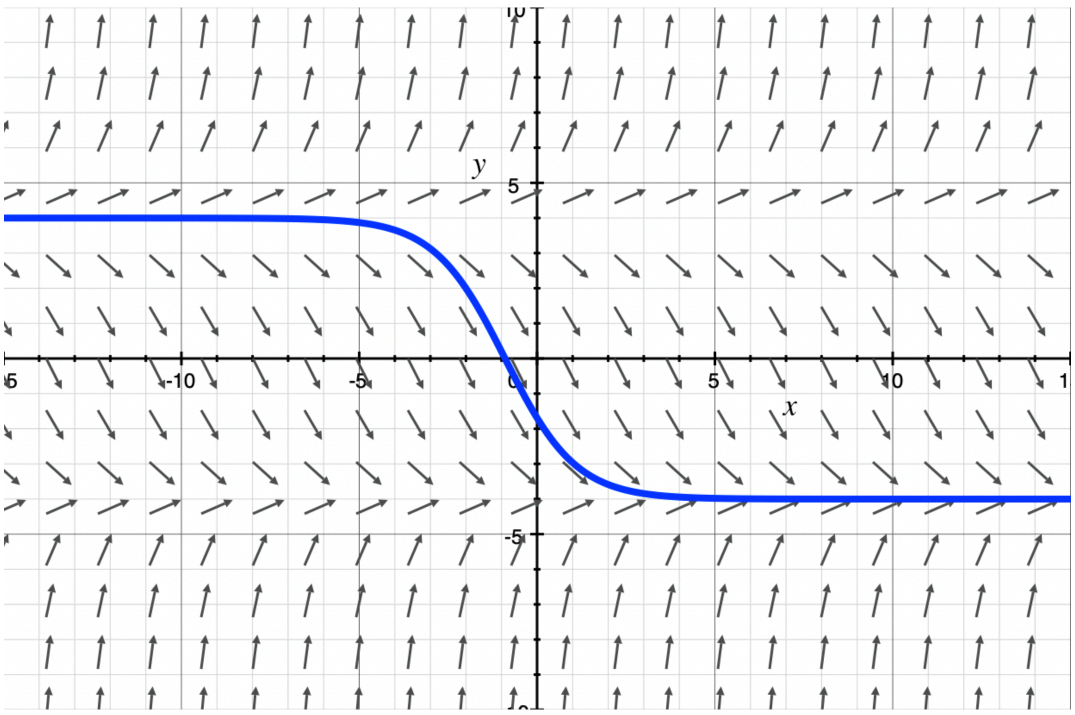
B 0

C 4

D ∞

Solution: C

We need to find the solution curve through $(-2,2)$, and consider its end behavior. In this slope field, it looks like there are horizontal "lines" at $y=4$ and $y=-4$, which represent horizontal asymptotes of the family of solution curves, so the curve through $(-2,2)$ would look like this:



Toward $-\infty$, the curve approaches the line $y=4$, so $\lim_{x\to-\infty} f(x)=4$.

■ 4. Which of the following is a solution to the differential equation $4y - xy' = -12$?

[A]	$y = 2x^4 - 3$	B	$y = 4x^2 - 3$
C	$y = 3x^2 - 4$	D	$y = 4x^3 - 12$

Solution: A

This problem is best solved by testing each answer choice. The derivative of answer choice A is

$$y' = 8x^3$$

Substitute the derivative into the given differential equation.

$$4y - xy' = -12$$

$$4(2x^4 - 3) - x(8x^3) = -12$$

$$8x^4 - 12 - 8x^4 = -12$$

$$-12 = -12$$

Because this is a true statement, answer choice A is a solution to $4y - xy' = -12$.

■ 5. Given that $y = g(x)$ is a particular solution to the differential equation $\frac{dy}{dx} = (x-2)(y^2-1)$, and the initial condition $f(1) = 2$, which of the following is the equation of the tangent line to $f(x)$ at the point $(1,2)$?

A $y - 2 = 3(x - 1)$

B $y = 2$

C $y - 3 = 2(x - 1)$

[D] $y - 2 = -3(x - 1)$

Solution: D

Evaluate the differential equation at the ordered pair $(1,2)$.

$$m = \frac{dy}{dx}(1-2)(2^2-1) = -3$$

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This gives a slope of -3 through the point $(1,2)$, so the tangent line equation at that point is $y-2=-3(x-1)$.

■ 6. Find the solution to the separable differential equation $\frac{du}{dt}=\frac{t^2-2t}{2u}$ when $u(0)=1$.

A $u=\sqrt{t^3-t^2+1}$

B $u=\sqrt{\frac{1}{3}t^3-t^2+1}$

C $u=\sqrt{t^3-t^2+\frac{1}{3}}$

D $u=\sqrt{\frac{1}{3}t^3-t^2+3}$

Solution: B

Separate variables, then integrate both sides of the equation.

$$2u\ du=(t^2-2t)\ dt$$

$$\int 2u\ du=\int t^2-2t\ dt$$

$$u^2=\frac{1}{3}t^3-t^2+C$$

Apply the initial condition $u(0)=1$ to find a value for C.

$$1^2=\frac{1}{3}(0)^3-(0)^2+C$$

$$C=1$$

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Substitute $C=1$, then solve the equation for u.

$$u^2=\frac{1}{3}t^3-t^2+1$$

$$u=\pm\sqrt{\frac{1}{3}t^3-t^2+1}$$

We can simplify further by substituting the initial condition $u(0)=1$.

$$1=\pm\sqrt{\frac{1}{3}(0)^3-(0)^2+1}$$

$$1=\pm\sqrt{1}$$

$$1=\pm 1$$

We can see that $1=+1$ and not $1=-1$, so we'll only take the positive square root for the solution, and the answer becomes

$$u=\sqrt{\frac{1}{3}t^3-t^2+1}$$

■ 7. A student solved the differential equation $\frac{dy}{dx}=y\sin x$ with the initial condition that $y=2$ when $x=\pi$. In which step did he make his first error?

Step 1: $\int y\ dy=\int \sin x\ dx$

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Step 2: $\frac{1}{2}y^2 = \cos y + C$

Step 3: Since $y(\pi) = 2$, $C = 3$

Step 4: $\frac{1}{2}y^2 = \cos y + 3$

Step 5: $y = \sqrt{2\cos y + 6}$

[A] Step 1   B Step 2

C Step 3   D Step 4

Solution: A

The student makes his first error in Step 1. If he'd separated variables correctly, he would have found

$$\int \frac{1}{y}\ dy = \int \sin x\ dx$$

The student goes also makes an error in Step 2 when he incorrectly integrates $\sin x$ to get $\cos y$. These two errors together lead to an entirely incorrect solution.

■ 8. Which of the following is the generic solution to the differential equation $\frac{dy}{dx} = 2y^2$?

A $y = -2x + C$ B $y = -\frac{1}{2x} + C$

C $y = \sqrt[3]{6x + C}$

D $y = -\dfrac{1}{2x + C}$

Solution: D

Separate variables, and the integrate both sides of the equation.

$$dy = 2y^2\ dx$$

$$\frac{1}{y^2}\ dy = 2\ dx$$

$$\int \frac{1}{y^2}\ dy = \int 2\ dx$$

$$-\frac{1}{y} = 2x + C$$

$$-1 = (2x + C)y$$

$$y = -\frac{1}{2x + C}$$

■ 9. The rate of change of a population of rabbits, $\dfrac{dR}{dt}$, is directly proportional to the population, R. If t is in terms of months, if there were initially 20 rabbits, and if there were 75 rabbits after 5 months, which equation represents the population of rabbits over time, $R(t)$?

A $R(t) = 2e^{11t}$

B $R(t) = 11t + 20$

C $R(t) = 20e^{0.2644t}$

D $R(t) = e^{0.8051t} + 19$

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Solution: C

Because growth rate is directly proportion to the population, we can write a differential equation, separate variables, then integrate both sides.

$$\frac{dR}{dt} = kR$$

$$dR = kR\ dt$$

$$\frac{1}{R}dR = k\ dt$$

$$\int \frac{1}{R}\ dR = \int k\ dt$$

$$\ln R = kt + C$$

We don't need $|R|$ on the left side of the equation, because the number of rabbits in the population will never be negative. Solve the equation for R.

$$e^{\ln R} = e^{kt+C}$$

$$R = e^{kt}e^{C}$$

$$R = Ce^{kt}$$

Use the given initial condition $R(0) = 20$

$$20 = Ce^{k(0)}$$

$$20 = C$$

Then substitute $R(5) = 75$ into $R = 20e^{kt}$, and solve for k.

$$75 = 20e^{k(5)}$$

$$3.75 = e^{5k}$$

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$$\ln(3.75) = \ln(e^{5k})$$

$$\ln(3.75) = 5k$$

$$k = 0.2\ln(3.75) \approx 0.2644$$

Then $R(t) = 20e^{0.2644t}$.

■ 10. Given the differential equation $\frac{dy}{dx} = y(x^2 - 1)$,

a. Sketch a slope field.

b. Find an expression for $\frac{d^2y}{dx^2}$ in terms of x and y.

c. Find the particular solution $y = f(x)$ that satisfies $f(3) = 2$.

Solution:

a. Create a table of values to determine the slope at set of points around the origin.

x	-2	-2	-2
y	1	0	-1
y’	0	-2	-4

x	-1	-1	-1
y	1	0	-1
y’	1	-1	-3

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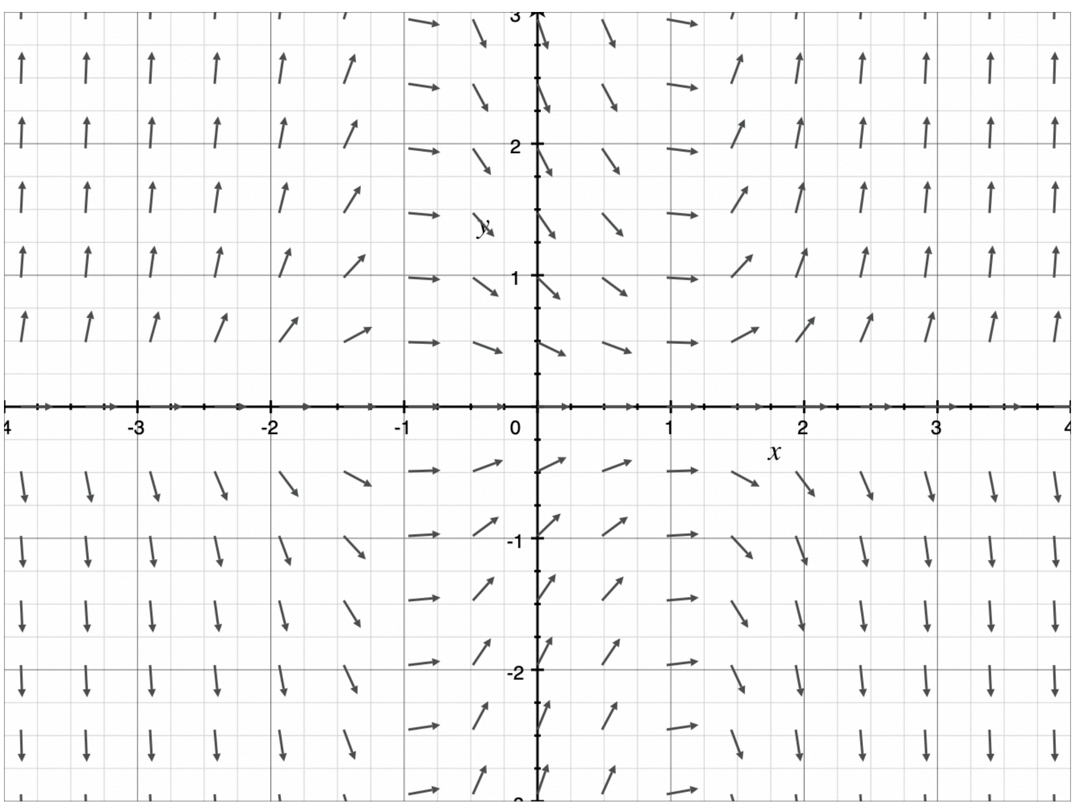
x	0	0	0
y	1	0	-1
y'	2	0	-2

x	1	1	1
y	1	0	-1
y'	3	1	-1

x	2	2	2
y	1	0	-1
y'	4	2	0

So the slope field starts to look something like this:



b. Use the product rule to find the derivative of the differential equation.

$$\frac{dy}{dx} = y(x^2 - 1)$$

$$\frac{d^2y}{dx^2}=\frac{dy}{dx}(x^2-1)+y(2x)$$

Substitute for $\frac{dy}{dx}$.

$$\frac{d^2y}{dx^2}=y(x^2-1)(x^2-1)+y(2x)$$

$$\frac{d^2y}{dx^2}=y(x^2-1)^2+2xy$$

c. To find the particular solution, start by separating variables and integrating both sides of the equation.

$$dy=y(x^2-1)\ dx$$

$$\int \frac{1}{y}\ dy=\int x^2-1\ dx$$

$$\ln|y|=\frac{x^3}{3}-x+C$$

Solve the equation for y.

$$e^{\ln|y|}=e^{\frac{x^3}{3}-x+C}$$

$$|y|=e^{\frac{x^3}{3}-x+C}$$

$$|y|=e^{\frac{x^3}{3}-x}e^C$$

$$y=Ce^{\frac{x^3}{3}-x}$$

Apply the initial condition $f(3)=2$.

$$2=Ce^{\frac{3^3}{3}-3}$$

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$$2 = Ce^6$$

$$C = \frac{2}{e^6}$$

Then the solution is

$$y = \frac{2}{e^6}e^{\frac{1}{3}x^3-x}$$

$$y = 2e^{-6}e^{\frac{1}{3}x^3-x}$$

$$y = 2e^{\frac{1}{3}x^3-x-6}$$

■ 11. Given the differential equation $\frac{dy}{dx} = x + 2y$,

a. Describe the region in the xy-plane where the slope field has positive slopes.

b. Given that $y = h(x)$ is a solution to this differential equation satisfying $h(3) = -2$, find the equation for the tangent line to $h(x)$ at the point $(-2,3)$, then use the tangent line to approximate $h(-2.1)$.

Solution:

a. To find regions containing only positive slope, we solve the inequality

$$0 < x + 2y$$

$$2y > -x$$

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$$y > -\frac{x}{2}$$

So all points above the line $y = -\frac{x}{2}$ will have positive slopes.

b. To find tangent line at $(-2,3)$, find the slope

$$\frac{dy}{dx} = m = -2 + 2(3) = 4$$

So the tangent line is $y - 3 = 4(x + 2)$. To approximate $h(-2.1)$, substitute $x = -2.1$.

$$y - 3 = 4(-2.1 + 2)$$

$$y - 3 = 4(-0.1)$$

$$y - 3 = -0.4$$

$$y = 3 - 0.4$$

$$y = 2.6$$

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