7.1 MODELING SITUATIONS WITH DIFFERENTIAL EQUATIONS

1. Match each situation with the differential equation that describes the relationship.

   a. The rate of change of a jogger's position, P (in meters), with respect to time t (in minutes), is directly proportional to the time t.

   $$\frac{dP}{dt} = kt$$

   b. The rate of change of a jogger's position, P (in meters), with respect to time t (in minutes), is directly proportional to the square root of the time t. $\frac{dP}{dt} = k\sqrt{t}$

   c. The rate of change of a jogger's position, P (in meters), with respect to time t (in minutes), is directly proportional to the square of the position P. $\frac{dP}{dt} = kP^2$

   d. The rate of change of a jogger's position, P (in meters), with respect to time t (in minutes), is directly proportional to the reciprocal of the square of time t. $\frac{dP}{dt} = \frac{k}{t^2}$

2. The amount of money in a savings account is measured by $B(t)$ over time t, in years. The rate of change of the amount of money is directly proportional to the square root of the amount in the account. At $t = 16$ years, the amount of money in the account is $\$250{,}000$ and is increasing at a rate of $\$750$ per year. Model this situation with a differential equation.

$$\frac{dB}{dt} = 1.5\sqrt{B}$$

7.2 VERIFYING SOLUTIONS FOR DIFFERENTIAL EQUATIONS

1. Match each differential equation with its possible solution.

$\frac{dy}{dx}=y$	$y=\sqrt{2x^3}$
$\frac{dy}{dx}=2xy$	$y=e^x$
$\frac{dy}{dx}=e^{x-y}$	$y=3e^{x^2}$
$\frac{dy}{dx}=\frac{3x^2}{y}$	$y=\ln(e^x+e^2-1)$

2. If $y=\frac{-3}{x^3}$, prove that the following equation is true for $x\neq 0$.

$$y''+\frac{3}{x}y'-\frac{3y}{x^2}=0$$

Find the first and second derivatives.

$$y'=\frac{9}{x^4}$$

$$y''=\frac{-36}{x^5}$$

Substitute the derivatives into the equation.

$$\frac{-36}{x^5}+\frac{3}{x}\left(\frac{9}{x^4}\right)-\frac{3\left(\frac{-3}{x^3}\right)}{x^2}=0$$

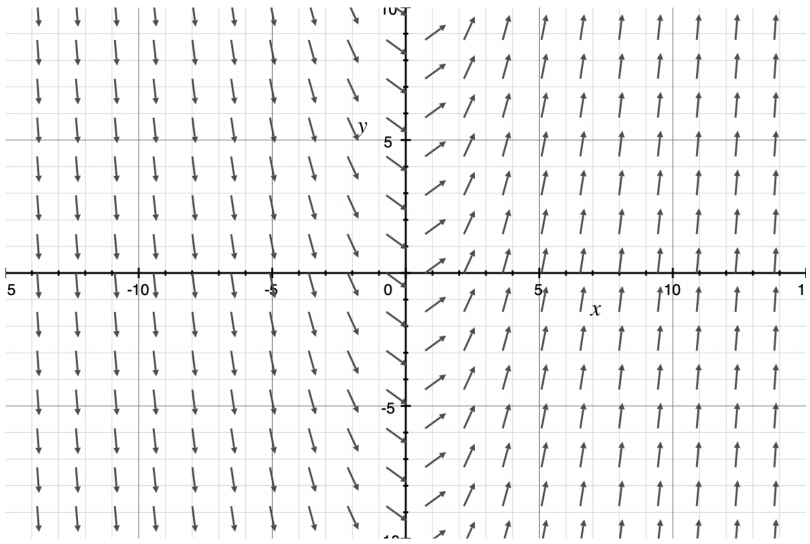
$$\frac{-36}{x^5}+\frac{27}{x^5}+\frac{9}{x^5}=0$$

$$0=0$$

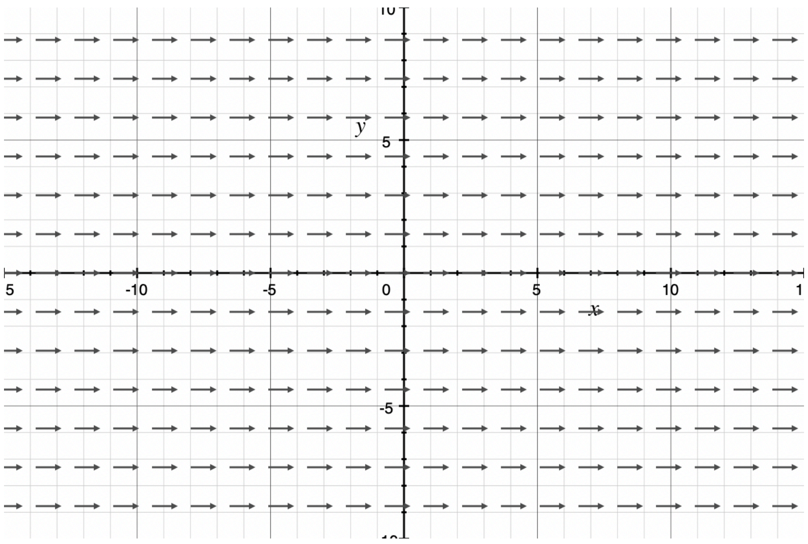
7.3 SKETCHING SLOPE FIELDS

1. Match each differential equation with the appropriate slope field.

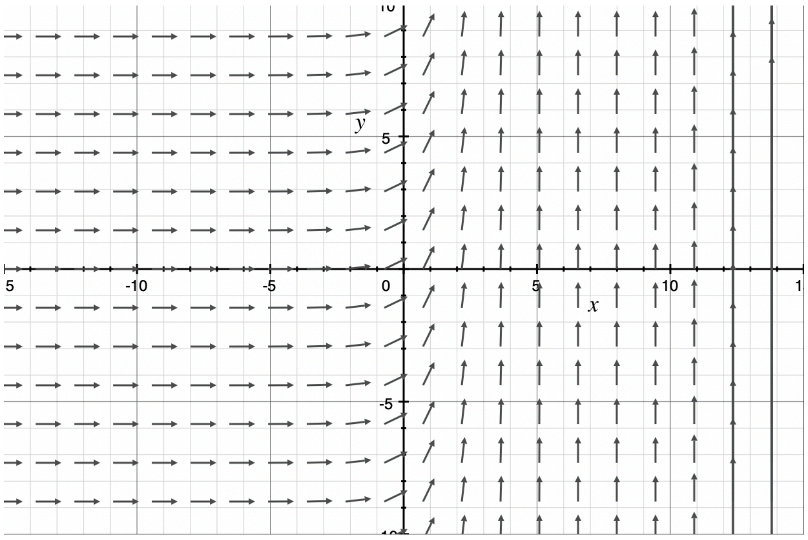
$$\frac{dy}{dx}=x$$



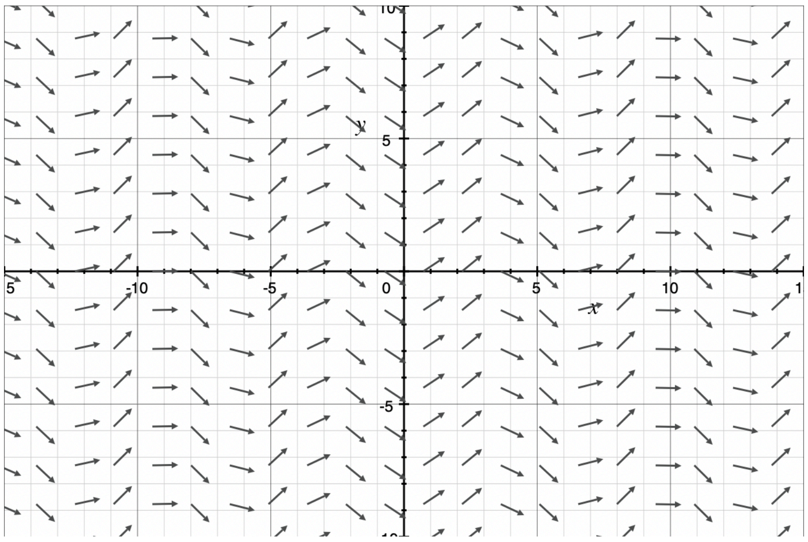
$$\frac{dy}{dx}=0$$



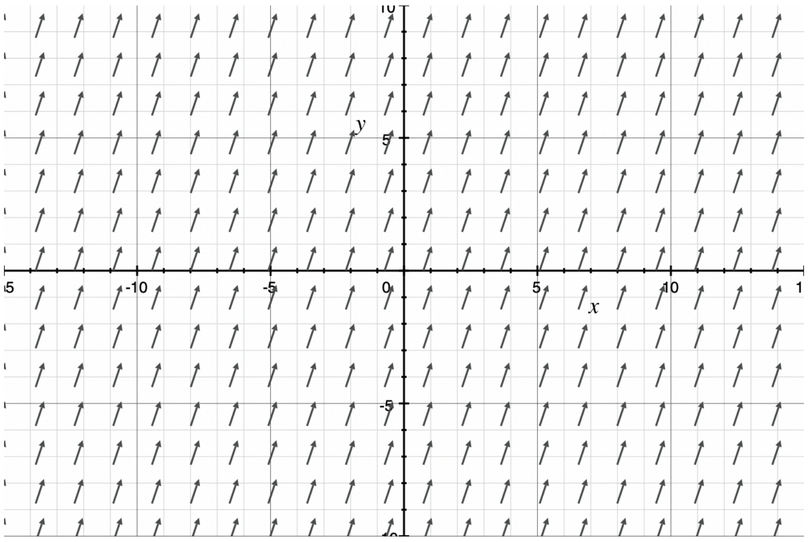
$$\frac{dy}{dx} = e^x$$



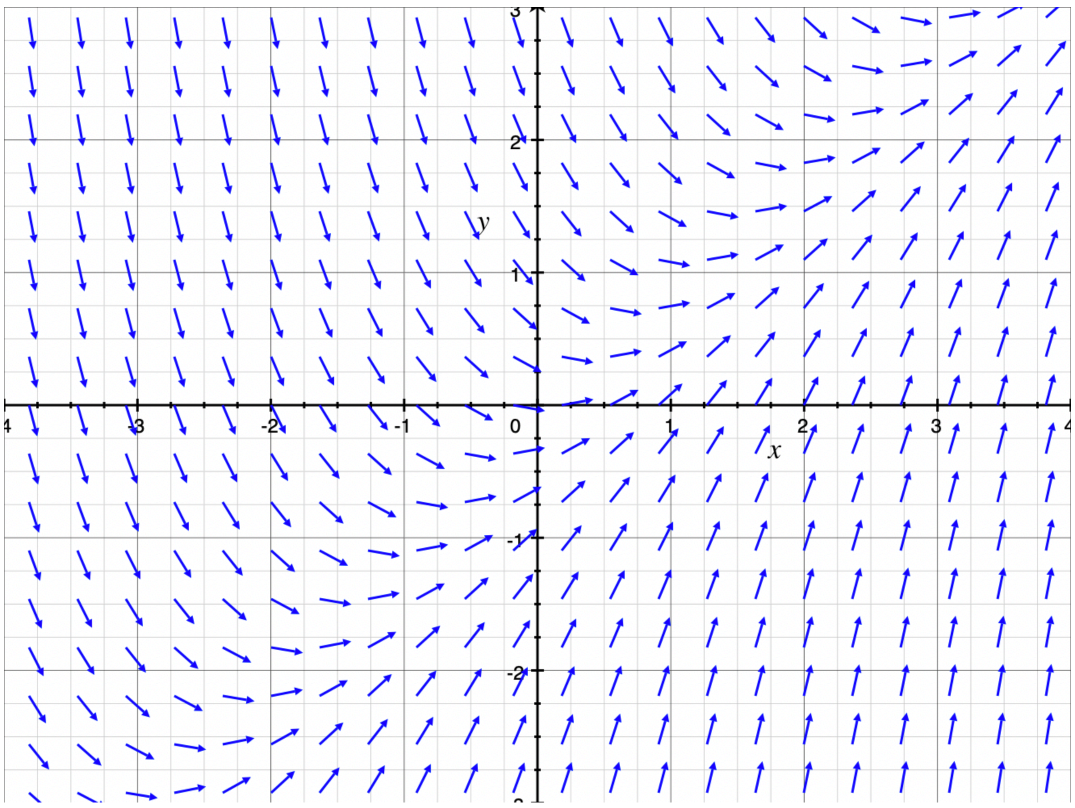
$$\frac{dy}{dx} = \sin x$$



$$\frac{dy}{dx} = 3$$



2. Sketch the slope field for $\dfrac{dy}{dx} = x - y$.



7.4 REASONING USING SLOPE

1. Mark each statement as true or false in regards to the slope field for $\frac{dy}{dx} = xy$.

 a. Every slope in Quadrant I will be positive.

  True. A positive x and y would give a positive product.

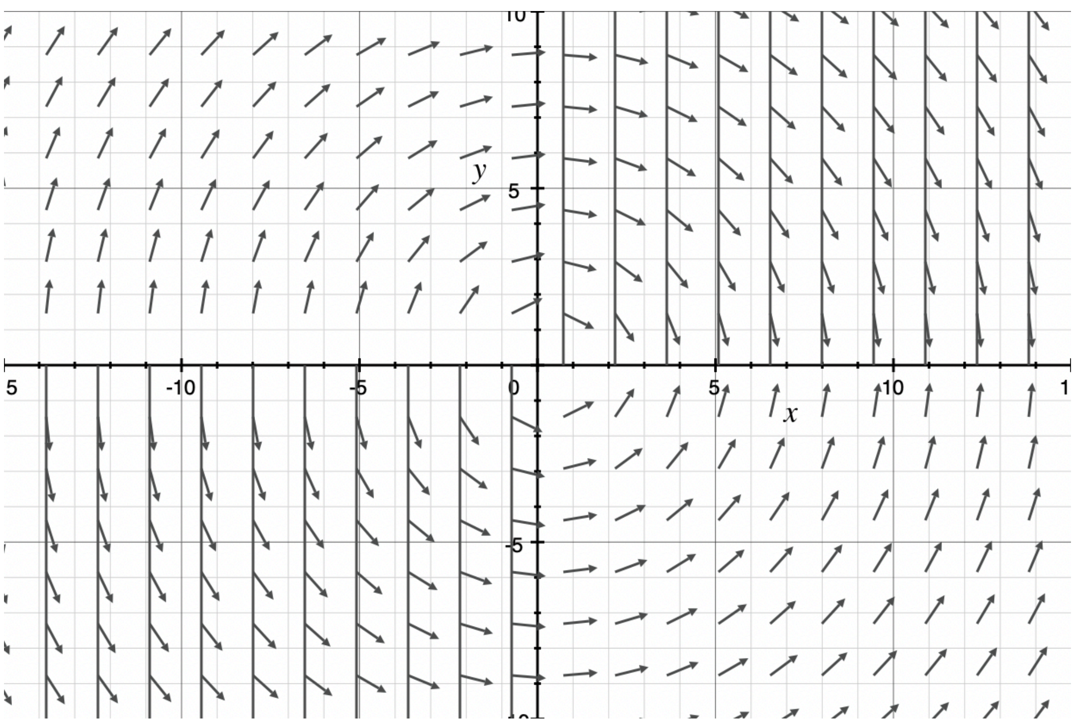
 b. Every slope in Quadrant III will be negative.

  False. A negative x and y would give a positive product.

 c. The slope along the x-and y-axes will be 0.

True, along the x-axis and y-axis, either y or x is equal to 0, so the product will be 0.

2. For the slope field below, write three observations about the graph, and then use your observations to circle the differential equation that represents the slope field.



$$\frac{dy}{dx} = y - x \qquad \frac{dy}{dx} = \frac{-x}{y} \qquad \frac{dy}{dx} = \frac{-y}{x} \qquad \frac{dy}{dx} = -x^3$$

Possible observations could include:

- All slopes are 0 when $x = 0$
- The slopes are symmetric about the x-axis and y-axis
- The slopes are positive in Quadrants II and IV, and negative in Quadrants I and III.

The only choice that matches these observations is $\frac{dy}{dx}=\frac{-x}{y}$.

3. For the slope field of $\frac{dy}{dx}=y^2(x-1)$, describe where the slopes would be negative.

The slopes would be negative when $x<1$.

7.6 FINDING GENERAL SOLUTIONS USING SEPARATION OF VARIABLES

1. Of the following differential equations, circle those that can be solved using separation of variables.

$$\frac{dy}{dx}=\frac{e^x}{\sqrt{y}} \qquad \frac{dy}{dx}=x+y$$

$$\frac{dy}{dx}+x=0 \qquad \frac{dy}{dx}=\frac{1+\sin x}{y^2-y}$$

2. Match each differential equation on the left to the associated general solution on the right.

$\dfrac{dy}{dx}=x^2y$	$\dfrac{y^3}{3}=\dfrac{x^2}{2}+C$
$\dfrac{dy}{dx}=\dfrac{x}{y^2}$	$\ln\lvert y\rvert=\ln\lvert x\rvert+C$
$\dfrac{dy}{dx}=\dfrac{y}{x}$	$\ln\lvert y\rvert=\dfrac{x^3}{3}+C$

3. The function $L(t)$ models the level of skill a person has acquired at time t, with $\dfrac{dL}{dt}=2(100-L)$ for $t\geq 0$ and $0\leq L<100$. Use separation of variables to show that $L(t)=100-e^{-2t-C}$.

Separate variables, then integrate both sides.

$$dL=2(100-L)\ dt$$

$$\frac{1}{100-L}\ dL=2\ dt$$

$$\int\frac{1}{100-L}\ dL=\int 2\ dt$$

$$-\ln(100-L)=2t+C$$

Solve for L.

$$\ln(100-L)=-2t-C$$

$$e^{\ln(100-L)}=e^{-2t-C}$$

$$100-L=e^{-2t-C}$$

$$L=100-e^{-2t-C}$$

7.7 FINDING PARTICULAR SOLUTIONS USING INITIAL CONDITIONS AND SEPARATION OF VARIABLES

1. For the following differential equations, use the initial condition to find the particular solution.

a. $\frac{dH}{dt} = H + 1$, $H(0) = 1$, $H(t) > 0$

$$\frac{1}{H+1}\ dH = 1\ dt \quad \to \quad \ln|H+1| = t + C$$

$\ln 2 = 0 + C$ gives $C = \ln 2$

$$\ln|H+1| = t + \ln 2 \quad \to \quad H + 1 = e^{t+\ln 2} \quad \to \quad H + 1 = e^t e^{\ln 2}$$

$$H + 1 = 2e^t \quad \to \quad H = 2e^t - 1$$

b. $\frac{dy}{dx} = \frac{x}{y}$, $y(1) = 0$

$$y\ dy = x\ dx \quad \to \quad \frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$

$0 = \frac{1}{2} + C$ gives $C = -\frac{1}{2}$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 - \frac{1}{2} \quad \to \quad y^2 = x^2 - 1 \quad \to \quad y = \pm\sqrt{x^2 - 1}$$

c. $\frac{dA}{dr} = 2\pi r$, $A(3) = 9\pi$

$$dA = 2\pi r\ dr \quad \to \quad A = \pi r^2 + C$$

$9\pi = \pi(3)^2 + C$ gives $C = 0$

$$A = \pi r^2$$

2. For $0 \le t \le 50$, the rate of change of lizards on a sidewalk in Florida at time t days can be modeled by $\frac{dA}{dt} = \frac{\sqrt{t}}{e^A}$. The number of lizards at $t = 0$ is 0. Find $A(t)$.

$$e^A \ dA = t^{\frac{1}{2}} \ dt \quad \to \quad e^A = \frac{2}{3}t^{\frac{3}{2}} + C$$

$e^0 = 0 + C$ gives $C = 1$

$$e^A = \frac{2}{3}t^{\frac{3}{2}} + 1 \quad \to \quad A = \ln\left(\frac{2}{3}t^{\frac{3}{2}} + 1\right)$$

7.8 EXPONENTIAL MODELS WITH DIFFERENTIAL EQUATIONS

1. If $\frac{dy}{dx} = \frac{1}{3}y$, complete the following table, rounding each value to the nearest thousandth.

Find the solution to the differential equation.

$$dy = \frac{1}{3}y \ dx \quad \to \quad \frac{1}{y} \ dy = \frac{1}{3} \ dx \quad \to \quad \ln|y| = \frac{1}{3}x + C$$

$$e^{\ln|y|} = e^{\frac{1}{3}x+C} \quad \to \quad |y| = e^{\frac{1}{3}x}e^C \quad \to \quad y = Ce^{\frac{1}{3}x}$$

Use the first row of the table to solve for C.

$$e = Ce^{\frac{1}{3}(0)} \quad \to \quad e = C$$

$$y = ee^{\frac{1}{3}x} \quad \to \quad y = e^{\frac{1}{3}x+1} \quad \to \quad \frac{dy}{dx} = \frac{1}{3}e^{\frac{1}{3}x+1}$$

Now we can use the equations for y and $\frac{dy}{dx}$ to complete the rest of the table.

x	y	dy/dx
0	e	0.906
2	5.294	1.765
1.828	6	1.667
5.671	18	6

2. A population triples every 50 years. The rate of the population's growth can be modeled by $\frac{dP}{dt} = kP$ for $0 \le t \le 500$, where k is some positive constant, t is measured in years, and P represents the population at time t. Solve for $P(t)$ if the population at $t = 0$ years is 100.

Separate variables, integrate, and simplify.

$$dP = kP\ dt \quad \to \quad \frac{1}{P}\ dP = k\ dt \quad \to \quad \ln|P| = kt + C$$

$$e^{\ln|P|} = e^{kt+C} \quad \to \quad |P| = e^{kt}e^{C} \quad \to \quad P = Ce^{kt}$$

$100 = Ce^0$ gives $C = 100$

$$P = 100e^{kt}$$

Because the population triples every 50 years, we can say $P(50) = 300$.

$$300 = 100e^{50k} \quad \to \quad 3 = e^{50k} \quad \to \quad \ln 3 = \ln(e^{50k})$$

$$\ln 3 = 50k \quad \to \quad k = \frac{\ln 3}{50}$$

$$P=100e^{\frac{\ln 3}{50}t} \quad \to \quad P=100e^{\ln(3^{\frac{1}{50}})t} \quad \to \quad P=100(3^{\frac{1}{50}})t$$