## 1.1 INTRODUCING CALCULUS: CAN CHANGE OCCUR AT AN INSTANT?

1. If average rate of change is given by $\Delta y/\Delta x$, what is the only value of either $\Delta y$ or $\Delta x$ at which average rate of change is undefined? $\Delta x = 0$

2. Use the table to find average rate of change at $x = 3$.

| x | 0 | 2 | 4 | 6 |
|---|---|---|---|---|
| f(x) | -7 | 6 | 4 | -1 |

$$\frac{f(x)-f(a)}{x-a} = \frac{f(4)-f(2)}{4-2} = \frac{4-6}{4-2} = \frac{-2}{2} = -1$$

3. Change each expression for average rate of change to an expression for instantaneous rate of change.

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

## 1.2 DEFINING LIMITS AND USING LIMIT NOTATION

1. True or False? The limit exists at $x = a$ and is a real number $L$ if the value of the function $f(x)$ gets closer and closer to $L$ as we can move as close to $a$ as we want, without arriving at exactly $a$.

## 1.3 ESTIMATING LIMIT VALUES FROM GRAPHS

1. The general limit exists at a point $x = c$ if

   a. the left-hand limit exists at $x = c$,

   b. the right-hand limit exists at $x = c$,

   c. those left- and right-hand limits are equal to one another

2. True or False? The general limit of $f(x) = |x| + 3$ exists at $x = -3$. Why or why not?

   True, because the left-hand limit exists and is equal to $6$, the right-hand limit exists and is equal to $6$, so the left- and right-hand limits are equal, and therefore the general limit exists.

3. Show that the limit $\lim_{x \to 2} \frac{1}{x - 2}$ does not exist.

   Use values on either side of $x = 2$, very close to $x = 2$, to determine how the function is behaving as $x \to 2$.

$$f(1.9999) = \frac{1}{1.9999 - 2} = \frac{1}{-0.0001} = -10{,}000$$

$$f(2.0001) = \frac{1}{2.0001 - 2} = \frac{1}{0.0001} = 10{,}000$$

   From the function's values around $x = 2$, we can tell that the function is tending toward $-\infty$ to the left of $x = 2$, and toward $\infty$ to the right of $x = 2$.

$$\lim_{x \to 2^-} \frac{1}{x - 2} = -\infty$$

$$\lim_{x \to 2^+} \frac{1}{x - 2} = \infty$$

Because the left- and right-hand limits aren't equal, the general limit of this function does not exist at $x = 2$.

## 1.4 ESTIMATING LIMIT VALUES FROM TABLES

1. Use the values in the table to write an expression for the limit of $f(x)$ at $x = 5$.

| x | 4.9998 | 4.9999 | 5 | 5.0001 | 5.0002 |
|---|---|---|---|---|---|
| f(x) | 5.9998 | 5.9999 | | 6.0001 | 6.0002 |

$$\lim_{x\to5} f(x) = 6$$

2. Use the values in the table to write an expression for the limit of $f(x)$ at $x = -2$.

| x | -3 | -2 | -1 | 0 | 1 |
|---|---|---|---|---|---|
| f(x) | 10 | | 8 | 7 | 6 |

$$\lim_{x\to-2} f(x) = 9$$

## 1.5 DETERMINING LIMITS USING ALGEBRAIC PROPERTIES OF LIMITS

1. The limit of a product is the _____product_____ of the limits.

2. The limit of a quotient is the _____quotient_____ of the limits.

3. Use properties of limits to evaluate each limit.

a. $\lim_{x\to4}\frac{1}{2}x^2-\sqrt{x}=\frac{1}{2}(4)^2-\sqrt{4}=6$

b. $\lim_{x\to2}3x^2-6x+4=3(2)^2-6(2)+4=4$

c. $\lim_{x\to\pi}\tan(3x)=\frac{\sin(3\pi)}{\cos(3\pi)}=0$

d. $\lim_{x\to-1}(x^2+3x+2)(x-1)=((-1)^2+3(-1)+2)(-1-1)=0$

## 1.6 DETERMINING LIMITS USING ALGEBRAIC MANIPULATION

1. Rewrite the function before evaluating the limit.

a. $\lim_{x\to3}\frac{x^2-7x+12}{x^2-9}=\lim_{x\to3}\frac{(x-4)(x-3)}{(x+3)(x-3)}=\lim_{x\to3}\frac{x-4}{x+3}=\frac{3-4}{3+3}=-\frac{1}{6}$

b. $\lim_{x\to9}\frac{x-9}{\sqrt{x}-3}=\lim_{x\to9}\frac{x-9}{\sqrt{x}-3}\left(\frac{\sqrt{x}+3}{\sqrt{x}+3}\right)=\lim_{x\to9}\frac{(x-9)(\sqrt{x}+3)}{x+3\sqrt{x}-3\sqrt{x}-9}$

$=\lim_{x\to9}\frac{(x-9)(\sqrt{x}+3)}{x-9}=\lim_{x\to9}\sqrt{x}+3=\sqrt{9}+3=3+3=6$

2. Use algebraic manipulation to evaluate the trig limit.

a. $\lim_{x\to0}\frac{7}{x\csc x}=\lim_{x\to0}\frac{7}{\frac{x}{\sin x}}=7\lim_{x\to0}\frac{\sin x}{x}=7(1)=7$

b. $\lim_{x\to0}\frac{\cos x\sin x}{x}=\lim_{x\to0}\cos x\cdot\lim_{x\to0}\frac{\sin x}{x}=\lim_{x\to0}\cos x\cdot(1)=\cos(0)=1$

## 1.7 SELECTING PROCEDURES FOR DETERMINING LIMITS

1. Choose the correct procedure, then evaluate the limit.

a. $\lim_{x \to -2} x^2 + 2x + 6$ (Substitution)

$$(-2)^2 + 2(-2) + 6 = 4 - 4 + 6 = 6$$

b. $\lim_{x \to 4} \frac{x^2 - 16}{x - 4}$ (Factoring)

$$\lim_{x \to 4} \frac{(x+4)(x-4)}{x-4} = \lim_{x \to 4} x + 4 = 4 + 4 = 8$$

c. $\lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h}$ (Conjugate method)

$$\lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h} \left( \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2} \right) = \lim_{h \to 0} \frac{(4+h) - 4}{h\sqrt{4+h} + 2h}$$

$$= \lim_{h \to 0} \frac{h}{h\sqrt{4+h} + 2h} = \lim_{h \to 0} \frac{1}{\sqrt{4+h} + 2} = \frac{1}{\sqrt{4+0} + 2} = \frac{1}{4}$$

## 1.8 DETERMINING LIMITS USING THE SQUEEZE THEOREM

1. The sine function oscillates back and forth on the interval $[-1,1]$.

2. The cosine function oscillates back and forth on the interval $[-1,1]$.

3. Which limits would we use to squeeze the limit?

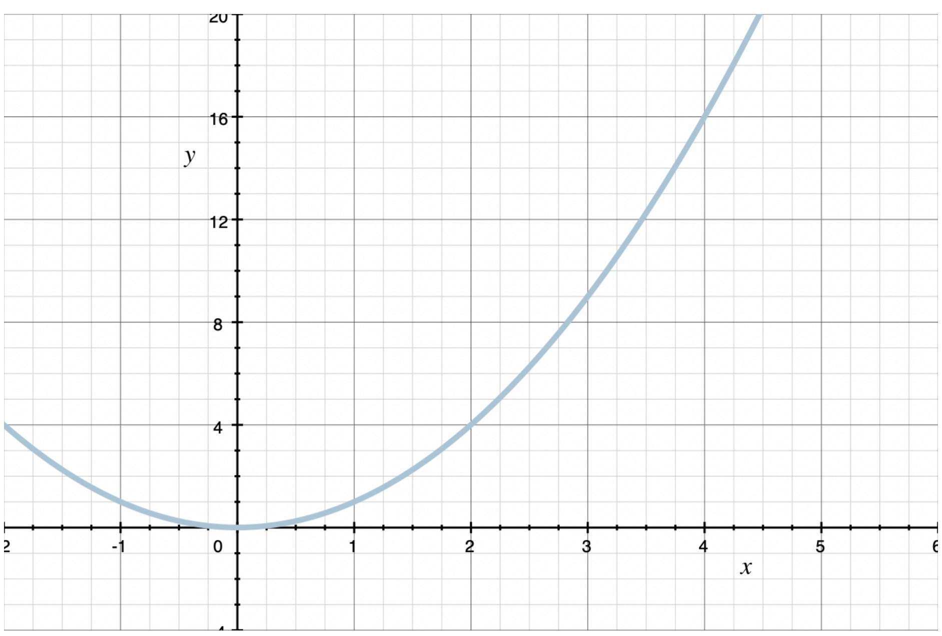
$$\lim_{x\to 0} -\frac{\sqrt{x}}{x} \le \lim_{x\to 0} \frac{\sqrt{x}\cos x}{x} \le \lim_{x\to 0} \frac{\sqrt{x}}{x}$$

## 1.9 CONNECTING MULTIPLE REPRESENTATIONS OF LIMITS

1. Show the following limit statement as an equation, a table, and a graph: "The limit as $x$ approaches $4$ of $x^2$ is $16$."

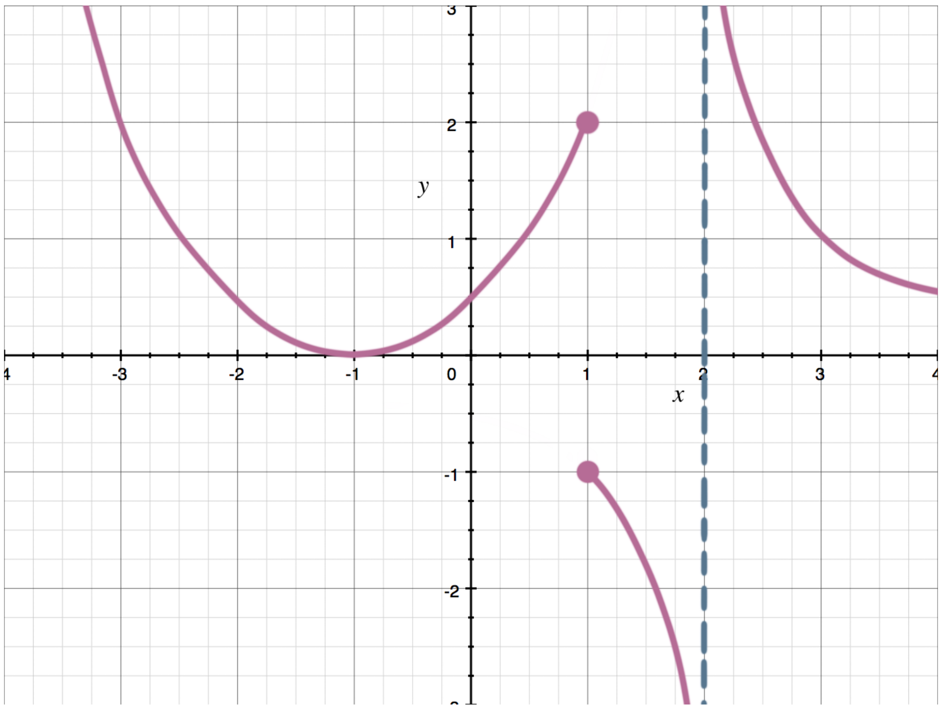
$$\lim_{x\to 4} x^2 = 16$$

| x | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| f(x) | 1 | 4 | 9 | 16 | 25 |




## 1.10 EXPLORING TYPES OF DISCONTINUITIES

1. Find and classify each discontinuity in the graph.




Jump discontinuity at $x = 1$, Infinite discontinuity at $x = 2$.

2. Infinite discontinuities are also called ____essential____ discontinuities.

3. Point discontinuities are also called ____removable____ discontinuities.

4. True or False? A discontinuity always exists at a cusp in the graph.
[False. A cusp is a sharp point in the graph. While a discontinuity *could* exist there, a cusp in the graph doesn't guarantee a discontinuity.]

## 1.11 DEFINING CONTINUITY AT A POINT

1. A function $f$ is continuous at $x = c$ if the following three conditions are met:

    1. $f(c)$ exists

    2. $\lim_{x\to c} f(x)$ exists

    3. $\lim_{x\to c} f(x) = f(c)$

2. What is the implication of the second condition?

The second condition, which is that $\lim_{x \to c} f(x)$ exists, implies that both the left- and right-hand limits exist at $x = c$, and that the values of the left- and right-hand limits are equal to one another at that point.

## 1.12 CONFIRMING CONTINUITY OVER AN INTERVAL

1. List six types of functions that are continuous everywhere in their domains.

Polynomial, Rational, Power, Exponential, Logarithmic, Trig

2. Which of the trig functions have their domains interrupted by horizontal asymptotes?

$\tan x$, $\csc x$, $\sec x$, $\cot x$

3. The domain of $\ln x$ is ______ $x > 0$ ______.

## 1.13 REMOVING DISCONTINUITIES

1. Redefine the function as a continuous piecewise function.

$$f(x) = \frac{x^2 + 11x + 28}{x + 4} = \frac{(x + 4)(x + 7)}{x + 4} = x + 7$$

The function has a removable discontinuity at $x=-4$. The value of the function there would be $f(x)=-4+7=3$, so plug the hole to remove the discontinuity by rewriting the function as

$$f(x)=\begin{cases}\frac{x^2+11x+28}{x+4} & x\neq -4\\ 3 & x=-4\end{cases}$$

2. Find the value of $k$ that makes the function continuous.

$$f(x)=\begin{cases}k\sqrt{x+1} & 0\le x\le 3\\ 5-x & 3<x\le 5\end{cases}$$

When $x=3$, we want the pieces to have equal value. So we set the pieces equal to one another, plug the break point $x=3$ into the equation, and then solve the equation for the unknown $k$.

$$k\sqrt{x+1}=5-x$$

$$k\sqrt{3+1}=5-3$$

$$k\sqrt{4}=2$$

$$2k=2$$

$$k=1$$

So $k=1$ is the value of the constant $k$ that will force the continuity of the function.

## 1.14 CONNECTING INFINITE LIMITS AND VERTICAL ASYMPTOTES

1. Where does the function have a vertical asymptote?

$$f(x) = \frac{1}{x-3}$$

At $x = 3$, since that's the value that makes the denominator equal $0$.

2. If $\lim_{x \to c^-} = \infty$ and $\lim_{x \to c^+} = -\infty$, does the general limit exist?

If both the left- and right-hand limits were $-\infty$, we could say the general limit was $-\infty$. And if both the left- and right-hand limits were $\infty$, we could say the general limit was $\infty$. But because the left- and right-hand limits aren't equal to $x = c$, the general limit does not exist.

## 1.15 CONNECTING LIMITS AT INFINITY AND HORIZONTAL ASYMPTOTES

1. Does the function have a horizontal asymptote? If so, where?

$$f(x) = \frac{1}{x-3}$$

Yes, at $y = 0$.

2. Describe the horizontal asymptote in each of the following cases.

N < D: If the degree of the numerator is less than the degree of the denominator, then the horizontal asymptote is given by $y = 0$.

N > D: If the degree of the numerator is greater than the degree of the denominator, then the function doesn't have a horizontal asymptote.

N = D: If the degree of the numerator is equal to the degree of the denominator, then the horizontal asymptote is given by the ratio of the coefficients on the highest-degree terms.

3. Find any horizontal asymptote that exists for each of the following functions.

a. $f(x) = \frac{x^3 - 2x^2}{x + 3}$ N > D, so the function doesn't have a horizontal asymptote

b. $g(x) = \frac{x + 3}{x^3 - 2x^2}$ N < D, so the function has a horizontal asymptote at $y = 0$

c. $h(x) = \frac{x^3 + 3}{x^3 - 2x^2}$ N = D, so the function has a horizontal asymptote at the ratio of coefficients, $y = 1/1 = 1$.

## 1.16 WORKING WITH THE INTERMEDIATE VALUE THEOREM (IVT)

1. True or False? The IVT states that, assuming $f(x)$ is continuous on $[a, b]$, if we can show that the function's value is negative at one side of the interval and positive at the other side, then that fact alone proves that the graph of the function must cross the $x$-axis at some point inside the interval.

2. Show that the function has at least one root in the interval $[-2,5]$.

$$f(x) = x^3 - 4x^2 + 7x + 1$$

The polynomial function is continuous for all real numbers.

$$f(-2) = (-2)^3 - 4(-2)^2 + 7(-2) + 1$$

$$f(-2) = -8 - 16 - 14 + 1$$

$$f(-2) = -37$$

and

$$f(5) = (5)^3 - 4(5)^2 + 7(5) + 1$$

$$f(5) = 125 - 100 + 35 + 1$$

$$f(5) = 61$$

Since we know that $f(x) = x^3 - 4x^2 + 7x + 1$ is continuous, and since the value of the function at the left side of the interval is negative (below the $x$-axis), and the value of the function at the right side of the interval is positive (above the $x$-axis), we know that there's at least one point $c$ at which the function will cross the $x$-axis.

$$-37 < f(c) < 61$$

Therefore, the function has at least one solution (root) in the interval $[-2,5]$.