■ 1. Find $\sum_{n=2}^{\infty} \frac{3}{4^n}$.

A $1$

$\boxed{\text{B}}$ $\frac{1}{4}$

C $\frac{3}{4}$

D Divergent

*Solution:* B

The formula for the sum of a Geometric series is $S = \frac{a_1}{1-r}$, so the sum is

$$S = \frac{\frac{3}{16}}{1-\frac{1}{4}} = \frac{1}{4}$$

■ 2. The nth term test can be used to determine divergence for which of the following series?

I. $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$

II. $\sum_{n=1}^{\infty} e^{-n}$

III. $\sum_{n=1}^{\infty} \frac{5n^2+3n}{2n^2-7}$

A III only

$\boxed{\text{B}}$ I and III

C II and III D I, II, and III

*Solution*: B

For the nth term test to determine divergence, $\lim_{n\to\infty} a_n \neq 0$.

I. $\lim_{n\to\infty} a_n \neq 0$

II. $\lim_{n\to\infty} a_n = 0$

III. $\lim_{n\to\infty} a_n \neq 0$

■ 3. Let $f$ be a positive, continuous, and decreasing function such that $a_n = f(x)$. If $\sum_{n=0}^{\infty} a_n$ converges to $k$, which one of the following must be true?

A $\int_1^\infty f(x)\ dx = k$

B $\int_1^\infty f(x)\ dx$ diverges

$\boxed{\text{C}}$ $\int_1^\infty f(x)\ dx$ converges

D None of the above

*Solution*: C

To prove convergence, the integral must converge to a finite value, but not necessarily to the same value as the infinite series.

■ 4. Which of the following is not a p-series?

$\boxed{\text{A}}$ $\sum_{n=1}^{\infty} \frac{5}{4^n}$

B $\sum_{n=1}^{\infty} \frac{1}{n}$

C $\sum_{n=1}^{\infty} \frac{5}{n^4}$

D $\sum_{n=1}^{\infty} n^{-3}$

*Solution*: A

Answer choice A is a geometric series, and the only series that's not a p-series.

■ 5. Given the series $\sum_{n=1}^{\infty} \frac{n^2(4^n)}{n!}$, which of the following inequalities would result if the ratio test was applied to determine that the series converges?

A $\lim_{n\to\infty} \frac{4}{n+1} < 1$

$\boxed{\text{B}}$ $\lim_{n\to\infty} \frac{4(n+1)}{n^2} < 1$

C $\lim_{n\to\infty} \frac{n+1}{4} > 1$

D $\lim_{n\to\infty} \frac{n^2}{4(n+1)} > 1$

*Solution*: B

The formula for the ratio test is $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ to determine convergence.

$$\lim_{n \to \infty} \left| \frac{\frac{4^{n+1}(n+1)^2}{(n+1)!}}{\frac{4^n n^2}{n!}} \right| \quad \to \quad \lim_{n \to \infty} \left| \frac{4^n 4^1 (n+1)^2 n!}{4^n n^2 (n+1)!} \right|$$

$$\lim_{n \to \infty} \left| \frac{4^1 (n+1)^2}{n^2 (n+1)} \right| \quad \to \quad \lim_{n \to \infty} \left| \frac{4(n+1)}{n^2} \right| < 1$$

## ■ 6. Which of the following conditionally converges?

A $\sum_{n=1}^{\infty} \frac{\sin \pi n}{n^3}$

B $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{\frac{3}{2}}}$

$\boxed{\text{C}}$ $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n}$

D $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n}$

*Solution*: C

Options A, B, and D would all converge without their alternator; therefore, they are all absolutely convergent. For answer choice C, $a_n = \frac{1}{n}$ would diverge without the alternator.

## ■ 7. Let $f(x) = x^2$. What is the approximation for $f(1.2)$ found by using the second-degree Taylor polynomial for $f$ about $x = 1$?

$\boxed{\text{A}}$ $1.44$

B $1.24$

C $0.44$

D $0.24$

*Solution*: A

For Taylor Polynomials, the formula is

$$f(x) \approx f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2$$

$$f(x) \approx P_2(x) = 1 + 2(x - 1) + (x - 1)^2$$

$$f(1.2) \approx P_2(1.2) = 1 + 2(1.2 - 1) + (1.2 - 1)^2 = 1.44$$

■ 8. What is the coefficient of the $x^3$ term of the Maclaurin polynomial for $f(x) = \dfrac{x^2}{e^x}$?

A $1$

$\boxed{\text{B}}$ $-1$

C $\dfrac{1}{3!}$

D $-\dfrac{1}{3!}$

*Solution*: B

The Maclaurin Polynomial for $f(x) = x^2e^{-x}$ is

$$f(x) \approx x^2 - x^3 + \frac{x^4}{2!} - \frac{x^5}{3!} + \ldots +$$

■ 9. The Maclaurin series for $\cos x$ is $\cos x = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots + \dfrac{(-1)^n x^{2n}}{(2n)!} + \ldots$. The continuous function $f$ is defined by $f(x) = x\cos x$. The function $f$ has derivative of all orders at $x = 0$.

a. Write the first four nonzero terms and the general term for the Taylor series for $f(x)$ centered at $c = 0$.

b. Use the seventh degree Taylor polynomial to approximate $f(0.3)$.

c. Use the Lagrange error bound to show that $|f(0.3) - P(0.3)| < \frac{1}{1{,}000}$.

d. Use the ratio test to find the interval of convergence for the Taylor series found in part a.

*Solution*:

a. The Taylor polynomial formula is

$$f(x) \approx f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots +$$

$$f(x) \approx (x-0) - \frac{3(x-0)^3}{3!} + \frac{5(x-0)^5}{5!} - \frac{7(x-0)^7}{7!} + \ldots + \frac{(-1)^n(2n+1)x^{2n+1}}{(2n+1)!} + \ldots$$

$$f(x) \approx x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n)!} + \ldots$$

b. Using $f(x) \approx P_7(x) = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!}$, calculate an approximation for $f(0.3)$.

$$f(0.3) \approx P_7(0.3) = 0.3 - \frac{0.3^3}{2!} + \frac{0.3^5}{4!} - \frac{0.3^7}{6!}$$

$$f(0.3) \approx 0.287$$

c. The maximum amount of error is contained the next term of the polynomial – often called the remainder term.

$$R_n(x) \le \left| \frac{Max}{(n+1)!}(x-a)^{n+1} \right|$$

$$R_7(0.3) \le \left| \frac{9}{(9)!}(0.3-0)^9 \right|$$

$$R_7(0.3) \le 0.000000000488$$

d. Applying the ratio test gives

$$\lim_{n\to\infty} \left| \frac{\frac{x^{2(n+1)+1}}{(2(n+1))!}}{\frac{x^{2n+1}}{(2n)!}} \right|$$

$$\lim_{n\to\infty} \left| \frac{x^{2n+3}(2n)!}{x^{2n+1}(2n+2)!} \right|$$

$$\lim_{n\to\infty} \left| \frac{x^{2n}x^3(2n)!}{x^{2n}x^1(2n+2)(2n+1)(2n)!} \right|$$

$$\lim_{n\to\infty} \left| \frac{x^2}{(2n+2)(2n+1)} \right| = 0$$

Therefore, the interval of the convergence is $(-\infty, \infty)$.