

10.1 DEFINING CONVERGENT AND DIVERGENT INFINITE SERIES

1. Match the n th partial sum of the series $\sum_{n=1}^{\infty} a_n$ on the left with the corresponding solution on the right.

$$S_n = 3 - \frac{5}{n^2}$$

$$S_n = \frac{n-2}{n+5}$$

$$S_n = 3 - n \cdot 2^{2-n}$$

$$S_n = 3 + \frac{5n^2}{n^2 + 7}$$

$$a_3 = \frac{1}{8}$$

$$\sum_{n=1}^3 a_n = \frac{7}{2}$$

$$a_1 = \frac{29}{8}$$

$$\sum_{n=1}^5 a_n = \frac{5,531}{720}$$

2. An infinite series of numbers converges to a real number S if and only if the limit of its sequence of partial sums exists and is equal to S .

3. Say whether the n th partial sum of the series $\sum_{n=1}^{\infty} a_n$ converges or diverges. If it converges, calculate the value of S .

$$S_n = \frac{5}{2} + \frac{3n^2}{2n^2 - 1}$$

converges, $S = 4$

$$S_n = \frac{2}{n} - 5n^2$$

diverges

$$S_n = \frac{n^2 + 1}{(n-3)(2n+1)}$$

converges, $S = \frac{1}{2}$

$$S_n = \frac{e^{2n}}{3 + e^{2n}}$$

converges, $S = 1$

$$S_n = \frac{e^n}{2n}$$

diverges

10.2 WORKING WITH GEOMETRIC SERIES

1. Circle any series below that are not geometric.

$$8 + 2 + \frac{2}{4} + \frac{2}{16} + \dots$$

$$\sum_{n=0}^{\infty} \frac{1}{n^4}$$

$$\sum_{n=0}^{\infty} \frac{3^n}{8^n}$$

$$\sum_{n=1}^{\infty} \frac{e^n}{2^{n-1}}$$

$$\sum_{n=0}^{\infty} \frac{n^4}{4^n}$$

$$8 - 12 + 36 - 48 + \dots$$

2. The geometric series $\sum_{n=0}^{\infty} ar^n$ converges to $\frac{a}{1-r}$ if $|r| < 1$, and diverges if $|r| \geq 1$.

3. $\sum_{n=0}^{\infty} \frac{3^n}{2^{2n+3}}$

a. True or False? The series given is geometric.

True. We can rewrite the series as

$$\sum_{n=0}^{\infty} \frac{3^n}{2^{2n} \cdot 8} = \sum_{n=0}^{\infty} \frac{1}{8} \cdot \frac{3^n}{2^{2n}} = \sum_{n=0}^{\infty} \frac{1}{8} \left(\frac{3}{4}\right)^n$$

b. True or False? The given series diverges.

False. $|r| = \frac{3}{4} < 1$. Therefore, the series converges.

c. If the series converges, find the sum.

$$\frac{a}{1-r} = \frac{\frac{1}{8}}{1-\frac{3}{4}} = \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{1}{2}$$

4. For each series, find the constant ratio r , determine whether the series converges or diverges, and find the sum.

Series	r	Converges or diverges	Sum
$\sum_{n=0}^{\infty} \frac{5}{10^{n+1}}$	$\frac{1}{10}$	Converges	$\frac{5}{9}$
$\sum_{n=1}^{\infty} 8^n$	8	Diverges	None
$\sum_{n=1}^{\infty} \frac{(-1)^n}{6^n}$	$-\frac{1}{6}$	Converges	$-\frac{1}{7}$
$\sum_{n=1}^{\infty} \frac{e^n}{5^{n-1}}$	$\frac{e}{5}$	Converges	$\frac{5e}{5-e}$
$\sum_{n=0}^{\infty} 3(0.75)^n$	0.75	Converges	12

$$\sum_{n=0}^{\infty} (\ln 2)^{-n}$$

$$\frac{1}{\ln 2}$$

Diverges

None

10.3 THE NTH TERM TEST FOR DIVERGENCE

1. If $\lim_{n \rightarrow \infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ will diverge.

2. Evaluate the limit of each series and circle the ones that diverge.

$$\sum_{n=1}^{\infty} \frac{2n!}{3n! + 1}$$

$$\lim_{n \rightarrow \infty} a_n = \frac{2}{3}$$

$$\sum_{n=1}^{\infty} \frac{(4n+5)(n+7)}{(n+1)^2(n-5)}$$

$$\lim_{n \rightarrow \infty} a_n = 0$$

$$\sum_{n=1}^{\infty} \frac{3n^3 - 12n^2 + 5n - 4}{2n^2 - 4n + 7}$$

$$\lim_{n \rightarrow \infty} a_n = \infty$$

$$\sum_{n=1}^{\infty} \frac{2^n - 5}{2^n}$$

$$\lim_{n \rightarrow \infty} a_n = 1$$

$$\sum_{n=1}^{\infty} \frac{\sin n}{n}$$

$$\lim_{n \rightarrow \infty} a_n = 0$$

$$\sum_{n=1}^{\infty} 1 - \frac{2}{\sqrt{n}}$$

$$\lim_{n \rightarrow \infty} a_n = 1$$

$$\sum_{n=1}^{\infty} \cos\left(\frac{2}{n}\right)$$

$$\lim_{n \rightarrow \infty} a_n = 1$$

$$\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n+7}$$

$$\lim_{n \rightarrow \infty} a_n = 0$$

10.4 INTEGRAL TEST FOR CONVERGENCE

1. Let f be a continuous, positive, decreasing function on $[1, \infty)$ and $a_n = f(n)$.

If $\int_1^{\infty} f(x) dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent. If $\int_1^{\infty} f(x) dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

2. Circle the series below to which the integral test can be applied.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$$\sum_{n=1}^{\infty} \frac{1}{3^n}$$

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

$$\sum_{n=0}^{\infty} \frac{5}{n^2 + 9}$$

$$\sum_{n=1}^{\infty} n^2$$

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

3. True or false? Explain your answer. Determine whether the series converges or diverges.

a. If $\sum_{n=1}^{\infty} 2^{-n}$, then $\lim_{a \rightarrow \infty} \frac{1}{2 \ln 2} - \frac{1}{2^a \ln 2}$. True, the series converges by the integral test.

$$\begin{aligned}\lim_{a \rightarrow \infty} \int_1^a 2^{-x} dx &= \lim_{a \rightarrow \infty} -\frac{1}{2^x \ln 2} \Big|_1^a \\ &= \lim_{a \rightarrow \infty} -\frac{1}{2^a \ln 2} - \left(-\frac{1}{2 \ln 2} \right) \\ &= \lim_{a \rightarrow \infty} \frac{1}{2 \ln 2} - \frac{1}{2^a \ln 2} = \frac{1}{2 \ln 2} - 0 = \frac{1}{2 \ln 2}\end{aligned}$$

- b. If $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$, then $\lim_{a \rightarrow \infty} \frac{1}{2} \ln a$. **False, the series diverges by the integral test.**

$$\lim_{a \rightarrow \infty} \int_1^a \frac{x}{x^2 + 1} dx$$

Let $u = x^2 + 1$ and $du = 2x dx$.

$$\begin{aligned}\lim_{a \rightarrow \infty} \int_1^a \frac{x}{x^2 + 1} dx &= \lim_{a \rightarrow \infty} \int \frac{1}{2} \cdot \frac{1}{u} du \\ &= \lim_{a \rightarrow \infty} \frac{1}{2} \ln |x^2 + 1| \Big|_1^a \\ &= \lim_{a \rightarrow \infty} \frac{1}{2} \ln |a^2 + 1| - \frac{1}{2} \ln 2 = \infty\end{aligned}$$

- c. If $\sum_{n=1}^{\infty} \frac{4}{(5n - 1)^2}$, then $\lim_{a \rightarrow \infty} \frac{1}{5} - \frac{4}{5(5a - 1)}$. **True, the series converges by the integral test.**

$$\lim_{a \rightarrow \infty} \int_1^a \frac{4}{(5x - 1)^2} dx$$

Let $u = 5x - 1$ and $du = 5 dx$.

$$\int \frac{4}{(5x-1)^2} dx = \int \frac{4}{5} \cdot \frac{1}{u^2} du = \frac{4}{5} \left(-\frac{1}{u} \right) = -\frac{4}{5u}$$

$$\begin{aligned} \lim_{a \rightarrow \infty} \int_1^a \frac{4}{(5x-1)^2} dx &= \lim_{a \rightarrow \infty} \left. -\frac{4}{5(5x-1)} \right|_1^a = \lim_{a \rightarrow \infty} \left(-\frac{4}{5(5a-1)} + \frac{4}{5(4)} \right) \\ &= \lim_{a \rightarrow \infty} \frac{1}{5} - \frac{4}{5(5a-1)} = \frac{1}{5} \end{aligned}$$

10.5 HARMONIC SERIES AND P-SERIES

1. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where p is any positive real number, is called a p -series. The p -series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$. If $p = 1$, we call the resulting series the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges.

2. Circle any series that converge by the p -series test.

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$$\sum_{n=1}^{\infty} n^{\frac{4}{3}}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{-3}}$$

$$\sum_{n=1}^{\infty} n^{-4}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$$

3. For each series, identify the value of p and use the p -series test to determine the convergence or divergence of each series.

Series	p	Converges or diverges
$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$	$p = \frac{1}{2}$	Diverges
$\sum_{n=1}^{\infty} n^{-\frac{\pi}{2}}$	$p = \frac{\pi}{2}$	Converges
$\sum_{n=4}^{\infty} \frac{1}{n^7}$	$p = 7$	Converges
$\sum_{n=1}^{\infty} \frac{1}{n^{0.8}}$	$p = 0.8$	Diverges
$\sum_{n=1}^{\infty} \frac{5}{n}$	$p = 1$	Diverges
$1 + \frac{1}{2\sqrt[3]{2}} + \frac{1}{3\sqrt[3]{3}} + \dots$	$p = \frac{4}{3}$	Converges

4. True or False? Explain your answers.

a. The series $\sum_{n=1}^{\infty} \frac{n}{n^{3p-2}}$ converges for $p > \frac{4}{3}$.

True. We can rewrite the series as

$$\sum_{n=1}^{\infty} \frac{n}{n^{3p-2}} = \sum_{n=1}^{\infty} \frac{1}{n^{3p-3}}$$

The p -series is convergent if $p > 1$, therefore

$$3p - 3 > 1 \quad \rightarrow \quad 3p > 4 \quad \rightarrow \quad p > \frac{4}{3}$$

b. The series $\sum_{n=1}^{\infty} \frac{1 - \sqrt[3]{n}}{n^2}$ converges with $p = 2$.

False. We can rewrite the series as

$$\sum_{n=1}^{\infty} \frac{1 - \sqrt[3]{n}}{n^2} = \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{\sqrt[3]{n}}{n^2} \right) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n^{\frac{5}{3}}}$$

The series converges because it's composed of two series with $p = 2$ and $p = \frac{5}{3}$.

c. The series $\sum_{n=1}^{\infty} \frac{n^2}{n^{2p+4}}$ is a harmonic series for $p = 1$.

False. We can rewrite the series as

$$\sum_{n=1}^{\infty} \frac{n^2}{n^{2p+4}} = \sum_{n=1}^{\infty} \frac{1}{n^{2p+2}}$$

The harmonic series is the special case where $p = 1$.

$$2p + 2 = 1 \quad \rightarrow \quad p = -\frac{1}{2}$$

10.6 COMPARISON TESTS FOR CONVERGENCE

1. Suppose $a_n \geq 0$, $b_n \geq 0$, and $a_n \leq b_n$ for all n . Then if $\sum b_n$ converges, then $\sum a_n$ converges, and if $\sum a_n$ diverges, then $\sum b_n$ diverges.

2. True or false? Explain your answer. Use the comparison test.

a. Since $0 < \frac{1}{n^5 + 2} < \frac{1}{n^5}$ for any $n \geq 1$, then the series $\sum_{n=1}^{\infty} \frac{1}{n^5 + 2}$ converges. True. Since we know that $0 < \frac{1}{n^5 + 2} < \frac{1}{n^5}$, then $\sum_{n=1}^{\infty} \frac{1}{n^5 + 2} < \sum_{n=1}^{\infty} \frac{1}{n^5}$. The series on the right is the convergent p -series with $p = 5 > 1$. Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{n^5 + 2}$ also converges by the comparison test.

b. Since $0 < \frac{5}{\sqrt{26n}} < \frac{5}{\sqrt{25n}} = \frac{1}{\sqrt{n}}$ for any $n \geq 1$, then the series $\sum_{n=1}^{\infty} \frac{5}{\sqrt{26n}}$ diverges. False. Since we know that $0 < \frac{5}{\sqrt{26n}} < \frac{5}{\sqrt{25n}} = \frac{1}{\sqrt{n}}$, then $\sum_{n=1}^{\infty} \frac{5}{\sqrt{26n}} < \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p -series with $p = \frac{1}{2} < 1$. But since $\sum_{n=1}^{\infty} \frac{5}{\sqrt{26n}} < \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, then the comparison test is inconclusive.

c. Since $0 < \frac{1 + \sin n}{10^n} < \frac{2}{10^n}$ for any $n \geq 1$, then $\sum_{n=0}^{\infty} \frac{1 + \sin n}{10^n}$ diverges. False. Since we know that $0 < \frac{1 + \sin n}{10^n} < \frac{2}{10^n}$, then $\sum_{n=0}^{\infty} \frac{1 + \sin n}{10^n} < \sum_{n=0}^{\infty} \frac{2}{10^n}$. The series on the right is a geometric series, which converges with $r = \frac{1}{10} < 1$. Therefore, by comparison test, the series $\sum_{n=0}^{\infty} \frac{1 + \sin n}{10^n}$ also converges.

3. Match each of the following comparison series to the correct series in the table below. Then determine whether each series from the second column converges or diverges and the reason for it. Finally, determine convergence or divergence of the original series.

$$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n \quad \sum_{n=1}^{\infty} e^{-n} \quad \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}} \quad \sum_{n=1}^{\infty} \frac{1}{n^4} \quad \sum_{n=1}^{\infty} \left(\frac{5}{8}\right)^n \quad \sum_{n=1}^{\infty} \frac{1}{n} \quad \sum_{n=1}^{\infty} \frac{1}{n}$$

Series	Comparison	Con/Diverges	Con/Diverges
$\sum_{n=1}^{\infty} \frac{e^{-n}}{n + \cos^2 n}$	$\sum_{n=1}^{\infty} e^{-n}$	Converges (integral test)	Converges
$\sum_{n=1}^{\infty} \frac{5^n}{8^n + 1}$	$\sum_{n=1}^{\infty} \left(\frac{5}{8}\right)^n$	Converges (geometric series)	Converges
$\sum_{n=1}^{\infty} \frac{\sqrt{n^6 - 1}}{n^7 + 7}$	$\sum_{n=1}^{\infty} \frac{1}{n^4}$	Converges (p -series test)	Converges
$\sum_{n=1}^{\infty} \frac{2^{\frac{1}{n}}}{n}$	$\sum_{n=1}^{\infty} \frac{1}{n}$	Diverges (harmonic series)	Diverges
$\sum_{n=1}^{\infty} \frac{n^2 - 8n}{n^3 + 2n + 1}$	$\sum_{n=1}^{\infty} \frac{1}{n}$	Diverges (harmonic series)	Diverges
$\sum_{n=1}^{\infty} \frac{n^2 - 1}{n^2 \sqrt{n}}$	$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$	Diverges (p -series test)	Inconclusive
$\sum_{n=1}^{\infty} \frac{n + 2^n}{n + 6^n}$	$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$	Converges (geometric series)	Converges

4. Suppose we have two series $\sum a_n$ and $\sum b_n$ with $a_n \geq 0$ and $b_n > 0$ for all n . $L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}$. If L is positive and finite, then either both series converge or both series diverge.

5. What steps do we take to apply limit comparison test?

1. Find a series whose convergence can be easily determined.
2. Find the limit using the formula $L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}$.
3. Determine whether L is positive.
4. Using the second series and the value of L , determine whether both series converge or both diverge.

6. For each series, find a comparison series whose convergence can be easily determined. Then find the limit and using the limit comparison test, determine whether each series converges or diverges.

Series	Second series	$\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$	Con/Diverges
$\sum_{n=2}^{\infty} \frac{5n^3 + n}{2\sqrt{n^9 + n^2}}$	$\sum_{n=2}^{\infty} \frac{1}{n^{\frac{3}{2}}}$	5	Both converge
$\sum_{n=1}^{\infty} \frac{1}{n^2 - 8}$	$\sum_{n=1}^{\infty} \frac{1}{n^2}$	1	Both converge
$\sum_{n=1}^{\infty} \frac{5 + 3^n}{2^n}$	$\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$	1	Both diverge
$\sum_{n=1}^{\infty} \frac{n + 5}{(n + 1)^3}$	$\sum_{n=1}^{\infty} \frac{1}{n^2}$	1	Both converge

$$\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n-1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}}$$

1

Both diverge

$$\sum_{n=1}^{\infty} \frac{1}{3n+5}$$

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$\frac{1}{3}$

Both diverge

10.7 ALTERNATING SERIES TEST FOR CONVERGENCE

- An alternating series $\sum (-1)^n a_n$ or $\sum (-1)^{n+1} a_n$ converges if $a_n > 0$ and if a_n is decreasing and $\lim a_n = 0$.
- For each series, determine whether or not $a_n > 0$, and whether or not a_n is increasing or decreasing. Then find the limit of a_n and determine whether the series converges or diverges.

Series	a_n	In/Decreasing	$\lim_{n \rightarrow \infty} a_n$	Con/Diverges
$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n+1}$	$\frac{1}{3n+1} > 0$	Decreasing	0	Converges
$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$	$\frac{1}{\sqrt{n}} > 0$	Decreasing	0	Converges
$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{2n^2+5}$	$\frac{n^2}{2n^2+5} > 0$	Increasing	$\frac{1}{2}$	Diverges
$\sum_{n=0}^{\infty} \frac{1}{(-1)^n(3^n+4^n)}$	$\frac{1}{3^n+4^n} > 0$	Decreasing	0	Converges

$$\sum_{n=2}^{\infty} \frac{(-1)^{n+1} e^n}{n^2} \quad \frac{e^n}{n^2} > 0 \quad \text{Increasing} \quad \infty \quad \text{Diverges}$$

3. Match the n th partial sum of the series on the left to the values of p on the right that cause the series to converge.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{p+7}} \quad \text{—————} \quad p > -6$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{5}{p}\right)^n \quad \text{—————} \quad p < 0$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} (3n)^p \quad \text{—————} \quad p > 5$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} (5p)^n \quad \text{—————} \quad p < \frac{1}{5}$$

10.8 RATIO TEST FOR CONVERGENCE

1. Let $\sum a_n$ be a decreasing series and $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$.

- if $L < 1$, then $\sum a_n$ converges.
- if $L > 1$ or if L is infinite, then $\sum a_n$ diverges.
- if $L = 1$, then there is no conclusion.

2. Complete the table.

Series	a_{n+1}	$\lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right $	Con/Diverges/Inconclusive
$\sum_{n=1}^{\infty} \frac{n^3}{(n+1)!}$	$\frac{(n+1)^3}{(n+2)!}$	0	Converges
$\sum_{n=1}^{\infty} \frac{n^{12}}{5^n}$	$\frac{(n+1)^{12}}{5^{n+1}}$	$\frac{1}{5}$	Converges
$\sum_{n=1}^{\infty} \frac{n(n+2)}{(n+4)(n+5)}$	$\frac{(n+1)(n+3)}{(n+5)(n+6)}$	1	Inconclusive
$\sum_{n=1}^{\infty} \frac{(5-n)7^n}{3 \cdot 5^n}$	$\frac{(4-n)7^{n+1}}{3 \cdot 5^{n+1}}$	$\frac{7}{5}$	Diverges
$\sum_{n=1}^{\infty} \frac{n^5}{2^{n^2}}$	$\frac{(n+1)^5}{2^{(n+1)^2}}$	0	Converges
$\sum_{n=1}^{\infty} \frac{n!}{7^n}$	$\frac{(n+1)!}{7^{n+1}}$	∞	Diverges

10.9 DETERMINING ABSOLUTE OR CONDITIONAL CONVERGENCE

- A series $\sum a_n$ is called absolutely convergent if $\sum |a_n|$ is convergent. If $\sum a_n$ is convergent and $\sum |a_n|$ is divergent, then the series is conditionally convergent.
- True or False? Explain your answer.

a. $\sum_{n=1}^{\infty} (-1)^{n+1} n^{-\frac{1}{5}}$ is absolutely convergent. **False.**

$\sum_{n=1}^{\infty} \left| (-1)^{n+1} n^{-\frac{1}{5}} \right| = \sum_{n=1}^{\infty} n^{-\frac{1}{5}}$. This series is divergent by the p -series test. $\sum_{n=1}^{\infty} (-1)^{n+1} n^{-\frac{1}{5}}$ is convergent by the alternating series test.

Therefore, $\sum_{n=1}^{\infty} (-1)^{n+1} n^{-\frac{1}{5}}$ is conditionally convergent.

b. $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ is conditionally convergent. **False.** $\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$,

and $|\sin n| \leq 1$, so we have $\frac{|\sin n|}{n^2} \leq \frac{1}{n^2}$. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p -series test, so $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$ also converges by the comparison test.

Therefore, $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ is absolutely convergent.

c. $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2 + 7}$ diverges. **True.** $\sum_{n=1}^{\infty} \left| \frac{(-1)^n n^2}{n^2 + 7} \right| = \sum_{n=1}^{\infty} \frac{n^2}{n^2 + 7}$. This series

diverges by the n th term test. $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2 + 7}$ diverges by alternating

series test. Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2 + 7}$ diverges.

10.10 ALTERNATING SERIES ERROR BOUND

1. Complete the sentence about the error of the approximation.

a. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^3 - 3}$ is approximated using the partial sum $\sum_{n=1}^{25} \frac{(-1)^{n+1}}{2n^3 - 3}$. The error bound is negative and so the approximation is an overestimate.

b. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n3^n}$ is approximated using the partial sum $\sum_{n=1}^{16} \frac{(-1)^{n+1}}{n3^n}$. The error bound is positive and so the approximation is an underestimate.

2. Find S_n as the approximation to the sum of the infinite series using the given number of terms. Set up an inequality to determine the maximum error for the approximation and find the maximum error. Then find an interval where the sum converges.

Series	S_n	Max error inequality	Closed interval
$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{3^n}$ (4)	$S_4 = \frac{14}{81}$	$ R_4 \leq \frac{5}{243}$	$0.152 \leq S \leq 0.193$
$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n!}$ (6)	$S_5 = \frac{11}{60}$	$ R_5 \leq \frac{1}{1,440}$	$0.183 \leq S \leq 0.184$
$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 5}{\sqrt{n}}$ (3)	$S_3 = 4.351$	$ R_3 \leq 2.5$	$1.851 \leq S \leq 6.851$
$\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{2n+5}}{\sqrt{n^2+1}}$ (10)	$S_{10} = -0.865$	$ R_{10} \leq 0.470$	$-1.335 \leq S \leq -0.395$
$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2}{3n^2}$ (5)	$S_5 = 0.559$	$ R_5 \leq 0.019$	$0.54 \leq S \leq 0.578$

3. Set up an inequality to determine the minimum number of terms needed to find the sum of the infinite series with an error less than 0.01. Solve each inequality to find the minimum number of terms needed.

a. $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n}$, then $|R_n| \leq \frac{1}{2n} \leq 0.01$.

$$|R_n| \leq \frac{1}{2n+2} \leq 0.01$$

$$2n+2 \geq 100 \rightarrow 2n \geq 98 \rightarrow n \geq \frac{98}{2} = 49$$

So 49 terms are needed.

b. $\sum_{n=0}^{\infty} \frac{5(-1)^n}{3n+1}$, then $|R_n| \leq \frac{5}{3n+4} \leq 0.01$.

$$|R_n| \leq \frac{5}{3(n+1)+1} = \frac{5}{3n+3+1} = \frac{5}{3n+4} \leq 0.01$$

$$\frac{3n+4}{5} \geq 100 \rightarrow 3n+4 \geq 500 \rightarrow 3n \geq 496$$

$$n \geq \frac{496}{3} \rightarrow n \geq 165.3$$

So 166 terms are needed.

c. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$, then $|R_n| \leq \frac{1}{n^4+1} \leq 0.01$.

$$|R_n| \leq \frac{1}{(n+1)^4} \leq 0.01$$

$$(n+1)^4 \geq 100 \rightarrow n+1 \geq \sqrt[4]{100} \rightarrow n+1 \geq 3.162$$

$$n \geq 2.162$$

So 3 terms are needed.

10.11 FINDING TAYLOR POLYNOMIAL APPROXIMATIONS OF FUNCTIONS

1. A power series about $x = a$ is a sum of constants multiplied by powers of $(x - a)$.
2. Match the following four expressions to its corresponding statement below.

a. Taylor polynomials have the following form.

$$P_n(x) = f(a) + \frac{f'(a)(x - a)}{1!} + \frac{f''(a)(x - a)^2}{2!} + \frac{f'''(a)(x - a)^3}{3!} + \dots$$

b. The coefficient of the n th degree term in a Taylor polynomial for a function f centered at $x = a$. $\frac{f^n(a)}{n!}$

c. Maclaurin polynomials are Taylor series that are used to approximate function values about. $x = 0$

d. The coefficient of the n th degree term in a Taylor polynomial for a function f centered at $x = 0$. $\frac{f^n(0)}{n!}$

3. Use the information given in the table below to write the first four nonzero terms of the Taylor series of each function.

	a	$f(a)$	$f'(a)$	$f''(a)$	$f'''(a)$
$f(x)$	5	$-\frac{1}{9}$	$\frac{2}{81}$	$-\frac{8}{729}$	$\frac{16}{2,187}$
$g(x)$	1	3	14	14	6
$h(x)$	2	11	15	22	12

$$f(x) = -\frac{1}{9} + \frac{2}{81}(x-5) - \frac{4}{729}(x-5)^2 + \frac{8}{6,561}(x-5)^3$$

$$g(x) = 3 + 14(x-1) + 7(x-1)^2 + (x-1)^3$$

$$h(x) = 11 + 15(x-2) + 11(x-2)^2 + 2(x-2)^3$$

4. Match each function on the left with the corresponding 3rd-degree Taylor polynomial.

$$f(x) = \frac{1}{x+1} \text{ at } x = 2 \quad e + \frac{e}{2}(x-2) + \frac{e}{8}(x-2)^2 + \frac{e}{48}(x-2)^3$$

$$f(x) = \sin 2x \text{ at } x = 0 \quad (x-2) - \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3$$

$$f(x) = e^{\frac{x}{2}} \text{ at } x = 3 \quad \frac{1}{3} - \frac{1}{9}(x-2) + \frac{1}{27}(x-2)^2 - \frac{1}{81}(x-2)^3$$

$$f(x) = \cos x \text{ at } x = \frac{\pi}{6} \quad 1 - \frac{1}{2}x^2$$

$$f(x) = \ln(x-1) \text{ at } x = 2 \quad \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{6}\right)^2 + \frac{1}{12}\left(x - \frac{\pi}{6}\right)^3$$

$$f(x) = \sin\left(x + \frac{\pi}{2}\right) \text{ at } x = 0 \quad 2x - \frac{4}{3}x^3$$

5. True or False? Explain your answer.

- a. If the n th derivative of f at $x = 0$ is given by $f^{(n)}(0) = \frac{(-1)^n(n+2)}{(n+1)3^n}$ for $n \geq 1$, then the coefficient for the term containing x^3 in the Maclaurin series is $-\frac{5}{108}$. **False.**

$$f(x) \approx f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$f'''(0) = f^{(3)}(0) = \frac{(-1)^3(3+2)}{(3+1)3^3} = \frac{-1 \cdot 5}{4 \cdot 27} = -\frac{5}{108}$$

$$\frac{f'''(0)}{3!} = \frac{-\frac{5}{108}}{1 \cdot 2 \cdot 3} = -\frac{5}{648}$$

- b. If the n th derivative of g at $x = 5$ is given by $g^{(n)}(5) = \frac{(-1)^n n!}{(n-3)2^n}$ for $n \geq 1$, then the coefficient for the term containing $(x-5)^4$ in the Taylor series is $\frac{1}{16}$. **True.**

$$g(x) = g(5) + \frac{g'(5)}{1!}(x-5) + \frac{g''(5)}{2!}(x-5)^2 + \frac{g'''(5)}{3!}(x-5)^3 + \frac{g^{(4)}(5)}{4!}(x-5)^4 + \dots$$

$$g^{(4)}(5) = g^{(4)}(5) = \frac{(-1)^4 4!}{(4-3)2^4} = \frac{2 \cdot 3 \cdot 4}{1 \cdot 16} = \frac{3}{2}$$

$$\frac{g^{(4)}(5)}{4!} = \frac{\frac{3}{2}}{4!} = \frac{1}{16}$$

- c. The coefficient for the term containing $(x+1)^5$ in the Taylor polynomial centered at $x = -1$, of $f(x) = x^5 + 2x^3 - x$, is 1. **True.**

$$p(x) = f(-1) + \frac{f'(-1)}{1!}(x+1) + \frac{f''(-1)}{2!}(x+1)^2 + \frac{f'''(-1)}{3!}(x+1)^3 + \frac{f^{iv}(-1)}{4!}(x+1)^4 + \frac{f^{v}(-1)}{5!}(x+1)^5$$

$$f'(x) = 5x^4 + 6x^2 - 1$$

$$f''(x) = 20x^3 + 12x$$

$$f'''(x) = 60x^2 + 12$$

$$f^{(4)}(x) = 120x$$

$$f^{(5)}(x) = 120$$

$$f^{(5)}(-1) = 120$$

$$\frac{f^{(5)}(-1)}{5!} = \frac{120}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 1$$

6. Approximate the following functions around $x = a$ and then approximate $f(a + 0.3)$.

$f(x)$	a	$f(a)$	$f'(a)$	$f''(a)$	$P_2(x)$	$f(a + 0.3)$
$1 + 5x + 2x^2 - 3x^3$	2	-5	-23	-32	$-5 - 23(x - 2) - 16(x - 2)^2$	-13.34
$\sin \frac{x}{2}$	π	1	0	$-\frac{1}{4}$	$1 - \frac{1}{8}(x - \pi)^2$	0.989
$\frac{1}{\sqrt{2x+9}}$	0	$\frac{1}{3}$	$-\frac{1}{27}$	$\frac{1}{81}$	$\frac{1}{3} - \frac{1}{27}x + \frac{1}{162}x^2$	0.323

10.12 LAGRANGE ERROR BOUND

1. Using the Lagrange error bound, write the appropriate meaning of each term.

$$|R_n| \leq \frac{M|x-a|^{n+1}}{(n+1)!}$$

R_n remainder (error)

x the given x -value

a the center point of the polynomial

n degree of the polynomial

M maximum value between a and x

2. Estimate the given functions using a Taylor polynomial about $x = a$, find the value of M , write the Lagrange error bound, and find the lowest degree polynomial that assures an error smaller than 0.001.

Function	a	M	$ R_n \leq \frac{M x-a ^{n+1}}{(n+1)!}$	n
$\cos(1.7\pi)$	2π	1	$ R_n(1.7\pi) \leq \left \frac{(1.7\pi - 2\pi)^{n+1}}{(n+1)!} \right $	5
$e^{0.90}$	1	e	$ R_n(0.90) \leq \left \frac{e(0.90 - 1)^{n+1}}{(n+1)!} \right $	2

$$\sin(-0.2) \quad 0 \quad 1 \quad |R_n(-0.2)| \leq \left| \frac{(-0.2)^{n+1}}{(n+1)!} \right| \quad 3$$

3. Use the Taylor polynomial for each of the following functions about $x = a$ to approximate the value of the function, then find the Lagrange bound for the error, and find the interval $[a, b]$.

a. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}, x = 0, \cos(-0.9) =$

$$\cos(-0.9) = 1 - \frac{(-0.9)^2}{2!} + \frac{(-0.9)^4}{4!} = 0.6223$$

$$|R_4(-0.9)| \leq \left| \frac{1 \cdot (-0.9)^6}{6!} \right| = 0.0007$$

$$0.6223 - 0.0007 \leq \cos(-0.9) \leq 0.6223 + 0.0007$$

$$0.622 \leq \cos(-0.9) \leq 0.623$$

b. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}, x = 0, e^{-1} =$

$$f(-1) = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} = \frac{3}{8} = 0.375$$

$$|R_4(-1)| \leq \left| \frac{e^0 (-1)^5}{5!} \right| = 0.008$$

$$0.375 - 0.008 \leq e^{-1} \leq 0.375 + 0.008$$

$$0.367 \leq e^{-1} \leq 0.383$$

c. $f(x) = 5 + 2(x - 3) - \frac{4}{2!}(x - 3)^2 + \frac{8}{3!}(x - 3)^3$, $|f^{(4)}(x)| \leq 6$, $f(3.2) =$

$$f(3.2) = 5 + 2(0.2) - 2(0.2)^2 + \frac{4}{3}(0.2)^3 = 5.331$$

$$|R_3(3.2)| \leq \left| \frac{6(0.2)^4}{4!} \right| = 0.004$$

$$5.331 - 0.004 \leq f(3.2) \leq 5.331 + 0.004$$

$$5.327 \leq f(3.2) \leq 5.335$$

d. $2 + \frac{1}{4}(x - 4) - \frac{\frac{1}{32}(x - 4)^2}{2!}$, $|f^{(3)}(x)| \leq \frac{3}{256}$, $x = 4$, $f(5) =$

$$f(5) = 2 + \frac{1}{4} - \frac{1}{64} = 2.219$$

$$|R_2(5)| \leq \left| \frac{\frac{3}{256}(1)^3}{3!} \right| = 0.002$$

$$2.219 - 0.002 \leq f(5) \leq 2.219 + 0.002$$

$$2.217 \leq f(5) \leq 2.221$$

10.13 RADIUS AND INTERVAL OF CONVERGENCE OF POWER SERIES

1. Complete the sequences.

a. A power series is a series of the form $\sum_{n=0}^{\infty} C_n(x - a)^n$, where n is a non-negative integer.

b. Use the ratio test for a power series $L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$, where $a_n =$
 $C_n(x - a)^n$.

- if L is infinite, then $R =$ 0.
- if $L = 0$, then $R =$ ∞ .
- if $L = K|x - a|$, then $R =$ $\frac{1}{K}$.

c. From the ratio test, the series converges when $L < 1$ and diverges when $L > 1$.

d. The interval of convergence is just the interval when $|x - a| < R$ or $a - R < x < a + R$.

2. For the following power series, write a_{n+1} , find the limit using the ratio test, and find the radius of convergence.

Power series	a_{n+1}	$\lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right $	R
$\sum_{n=1}^{\infty} \frac{(-1)^n(x+2)^n}{3^n}$	<u>$\frac{(-1)^{n+1}(n+1)(x+2)^{n+1}}{3^{n+1}}$</u>	<u>$\frac{1}{3} x+2$</u>	<u>3</u>

$$\sum_{n=1}^{\infty} \frac{(x-5)^{2n}}{(n+1)(-2)^n} \quad \frac{(x-5)^{2n+2}}{(n+2)(-2)^{n+1}} \quad \left| \frac{(x-5)^2}{2} \right| \quad \sqrt{2}$$

$$\sum_{n=0}^{\infty} n^2 x^n \quad (n+1)^2 x^{n+1} \quad |x| \quad 1$$

$$\sum_{n=0}^{\infty} (n+1)!(x-5)^n \quad (n+2)!(x-5)^{n+1} \quad \infty \quad 0$$

$$\sum_{n=1}^{\infty} \frac{(x+3)^n(-2)^n}{(2n+1)} \quad \frac{(x+3)^{n+1}(-2)^{n+1}}{(2n+3)} \quad 2|x+3| \quad \frac{1}{2}$$

$$\sum_{n=0}^{\infty} \frac{n(x-4)^n}{(3n+1)!} \quad \frac{(n+1)(x-4)^{n+1}}{(3n+4)!} \quad 0 \quad \infty$$

3. True or False? Explain your answer.

a. The interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n}$ is $4 \leq x < 6$.

True. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x-5|$. The series converges when

$|x-5| < 1$, so $-1 < x-5 < 1$ and $4 < x < 6$.

At $x = 4$, $\sum_{n=1}^{\infty} \frac{(4-5)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. This is the alternating harmonic series, which converges.

At $x = 6$, $\sum_{n=1}^{\infty} \frac{(6-5)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$. This is the harmonic series, which diverges.

Therefore, the interval of convergence is $4 \leq x < 6$.

b. The interval of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{5^n}$ is $-5 \leq x < 5$.

False. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x}{5} \right|$. The series converges when $\left| \frac{x}{5} \right| < 1$, so $-1 < \frac{x}{5} < 1$ and $-5 < x < 5$.

At $x = -5$, $\sum_{n=0}^{\infty} \frac{(-5)^n}{5^n} = \sum_{n=0}^{\infty} (-1)^n$. By the geometric series test, the series diverges.

At $x = 5$, $\sum_{n=0}^{\infty} \frac{5^n}{5^n} = \sum_{n=0}^{\infty} 1^n$. By the geometric series test, the series diverges.

Therefore, the interval of convergence is $-5 < x < 5$.

c. The interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(x+5)^n}{(n+1)(-5)^n}$ is $-10 < x \leq 0$.

True. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x+5}{5} \right|$. The series converges when $\left| \frac{x+5}{5} \right| < 1$, so $-1 < \frac{x+5}{5} < 1$, $-5 < x+5 < 5$, and $-10 < x < 0$.

At $x = -10$, $\sum_{n=1}^{\infty} \frac{(-10+5)^n}{(n+1)(-5)^n} = \sum_{n=1}^{\infty} \frac{1}{n+1}$. By the integral test, the series diverges.

At $x = 0$, $\sum_{n=1}^{\infty} \frac{(0+5)^n}{(n+1)(-5)^n} = \sum_{n=1}^{\infty} \frac{5^n}{(n+1)(-5)^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)}$. By the alternating series test, the series converges.

Therefore, the interval of convergence is $-10 < x \leq 0$.

10.14 FINDING TAYLOR OR MACLAURIN SERIES OF A FUNCTION

1. Match the function on the left with the corresponding power series.

Function	Series
$\cos x$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$
e^x	$1 + x + x^2 + x^3 + x^4 + \dots$
$\frac{1}{1-x}$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$
$\sin x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

2. True or False? Explain your answers.

a. The first three non-zero terms of the Maclaurin series for

$$f(x) = \frac{\sin(-x^2)}{x^2} \text{ are } -1 + \frac{x^4}{3!} - \frac{x^8}{5!}. \text{ True.}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\begin{aligned} \sin(-x^2) &= (-x^2) - \frac{(-x^2)^3}{3!} + \frac{(-x^2)^5}{5!} - \frac{(-x^2)^7}{7!} + \dots \\ &= -x^2 + \frac{x^6}{3!} - \frac{x^{10}}{5!} + \frac{x^{14}}{7!} + \dots \end{aligned}$$

$$\frac{\sin(-x^2)}{x^2} = -1 + \frac{x^4}{3!} - \frac{x^8}{5!} + \frac{x^{12}}{7!} + \dots$$

- b. The first three non-zero terms of the Maclaurin series for $f(x) = \frac{3}{5-x}$ are $1 + \frac{3}{5}x + \frac{9}{25}x^2$. **False.**

$$\frac{3}{5-x} = \frac{\frac{3}{5}}{\frac{5-x}{5}} = \frac{\frac{3}{5}}{1-\frac{x}{5}} = \frac{3}{5} \left(\frac{1}{1-\frac{x}{5}} \right)$$

$$\frac{1}{1-\frac{x}{5}} = 1 + \frac{x}{5} + \left(\frac{x}{5}\right)^2 + \left(\frac{x}{5}\right)^3 + \dots = 1 + \frac{x}{5} + \frac{x^2}{25} + \frac{x^3}{125} + \dots$$

$$\frac{3}{5} \left(\frac{1}{1-\frac{x}{5}} \right) = \frac{3}{5} \left(1 + \frac{x}{5} + \frac{x^2}{25} + \frac{x^3}{125} + \dots \right) = \frac{3}{5} + \frac{3x}{25} + \frac{3x^2}{125} + \frac{3x^3}{625} + \dots$$

- c. The first three non-zero terms of the Maclaurin series for $f(x) = xe^{-x^3}$ are $1 - x^3 + \frac{x^6}{2!}$. **False.**

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{-x^3} = 1 + (-x^3) + \frac{(-x^3)^2}{2!} + \frac{(-x^3)^3}{3!} + \dots = 1 - x^3 + \frac{x^6}{2!} - \frac{x^9}{3!} + \dots$$

$$xe^{-x^3} = x \left(1 - x^3 + \frac{x^6}{2!} - \frac{x^9}{3!} + \dots \right) = x - x^4 + \frac{x^7}{2!} - \frac{x^{10}}{3!} + \dots$$

3. Match the following Maclaurin series on the left with the corresponding expression for $f(x)$ on the right.

Maclaurin series

Function

$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots + \frac{x^{2n}}{(2n)!}$	$\frac{1}{1 + 8x^3}$
$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+1)!}$	$-x^2 \sin(-x)$
$1 - 8x^3 + 64x^6 + \dots + (-2x)^{3n}$	$\cos 2x$
$1 - 2x^2 + \frac{2x^4}{3} + \dots + \frac{(-4)^n x^{2n}}{(2n)!}$	$\frac{1}{2}(e^x + e^{-x})$
$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$	$x^2 \cos x^2$
$\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n)!}$	e^{-x}

4. Write the function that we need to use to evaluate each Maclaurin series.

	$f(x)$	Value
$1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} + \dots + \frac{(-1)^n \pi^{2n}}{(2n)!}$	$\cos x$	$\cos \pi = 1$
$1 + 3 + \frac{9}{2!} + \dots + \frac{3^n}{n!}$	e^x	e^3
$2 - \frac{2^3}{2!} + \frac{2^5}{5!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$\sin x$	$\sin 2$
$1 + 2 + 2^2 + 2^3 + \dots + 2^n$	$\frac{1}{1-x}$	-1

5. Use the Maclaurin series to evaluate each of the following functions.

a. $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$, then $f(\sqrt{\ln 5}) = \underline{\quad 5 \quad}$.

b. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$, then $f\left(\sqrt{\frac{\pi}{2}}\right) = \underline{\quad 0 \quad}$.

c. $\sum_{n=0}^{\infty} (-1)^n x^{3n}$, then $f(\sqrt[3]{7}) = \underline{\quad \frac{1}{8} \quad}$.

10.15 REPRESENTING FUNCTIONS AS POWER SERIES

1. Match each of the following power series of $f(x)$ with the corresponding power series of $f''(x)$ and then evaluate $f''(0)$.

Power series of $f(x)$

Power series of $f''(x)$

$f''(0)$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3x^{2n}}{(2n)!}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n-1}}{(n-1)!}$$

$$\underline{\quad -1 \quad}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)!}$$

$$\sum_{n=0}^{\infty} \frac{(2n+2)(2n+1)x^{2n}}{n!}$$

$$\underline{\quad 2 \quad}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{3(2n+1)!}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3x^{2n-2}}{(2n-2)!}$$

$$\underline{\quad 3 \quad}$$

$$\sum_{n=0}^{\infty} \frac{x^{2n+2}}{n!}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-1}}{3(2n-1)!}$$

$$\underline{\quad -\frac{1}{3} \quad}$$

2. True or False? Explain your answers.

a. $f(x) = \sum_{n=0}^{\infty} \frac{n+1}{3^{n+1}} x^n$, then $\int_0^1 f(x) dx = 3$. **False.**

$$\begin{aligned} \int_0^1 f(x) dx &= \int_0^1 \sum_{n=0}^{\infty} \frac{n+1}{3^{n+1}} x^n dx = \sum_{n=0}^{\infty} \int_0^1 \frac{n+1}{3^{n+1}} x^n dx \\ &= \sum_{n=0}^{\infty} \frac{n+1}{3^{n+1}} \cdot \frac{x^{n+1}}{n+1} \Big|_0^1 = \sum_{n=0}^{\infty} \frac{x^{n+1}}{3^{n+1}} \Big|_0^1 = \sum_{n=0}^{\infty} \frac{1^{n+1}}{3^{n+1}} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{n+1} = \sum_{n=0}^{\infty} \frac{1}{3} \left(\frac{1}{3}\right)^n \end{aligned}$$

This is the geometric series with the first term $\frac{1}{3}$ and common ratio $\frac{1}{3}$, so $\frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$.

b. $f(x) = \sum_{n=0}^{\infty} \frac{n+2}{4^n} x^{n+1}$, then $\int_0^1 f(x) dx = \frac{4}{3}$. **True.**

$$\begin{aligned} \int_0^1 f(x) dx &= \int_0^1 \sum_{n=0}^{\infty} \frac{n+2}{4^n} x^{n+1} dx = \sum_{n=0}^{\infty} \int_0^1 \frac{n+2}{4^n} x^{n+1} dx \\ &= \sum_{n=0}^{\infty} \frac{n+2}{4^n} \cdot \frac{x^{n+2}}{n+2} \Big|_0^1 = \sum_{n=0}^{\infty} \frac{x^{n+2}}{4^n} \Big|_0^1 = \sum_{n=0}^{\infty} \frac{1}{4^n} \end{aligned}$$

This is the geometric series with the first term 1 and common ratio $\frac{1}{4}$, so $\frac{1}{1 - \frac{1}{4}} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$.