■ 1. The population of box turtles in a wooded area is given by $R(t) = 3{,}000 - 1{,}200e^{-0.12t}$, where t is measured in years. What would the meaning of $R'(4) = 89.104$ be in the context of the problem?

A There are an average of 89.104 turtles in the wooded area over the first 4 years.

[✓ B] When $t = 4$ years, the number of turtles in the wooded area is increasing by 89.104 turtles per year.

C In year 4, approximately 89 more turtles will come into the wooded area.

D There will be a total of approximately 89 turtles in the forest after 4 years.

Solution: B

Derivatives give you a rate of change at one moment in time.

■ 2. Given the function $h(x) = 2xe^{x-4}$, find an equation for the tangent line to the curve when $x = 5$.

A $y - 5 = 12e(x - 10e)$

B $y - 10e = 2e(x - 5)$

C $y - 5 = 2e(x - 10e)$

[✓ D] $y - 10e = 12e(x - 5)$

Solution: D

At $x = 5$, the original function gives

$$h(5) = 2(5)e^{5-4}$$

$$h(5) = 10e$$

Now find the derivative using the product rule.

$$h'(x) = (2)(e^{x-4}) + (2x)(e^{x-4})(1)$$

$$h'(x) = 2e^{x-4} + 2xe^{x-4}$$

Substitute $x = 5$ into the derivative to find the slope at that point.

$$h'(5) = 2e^{5-4} + 2(5)e^{5-4}$$

$$h'(5) = 2e + 10e$$

$$h'(5) = 12e$$

Then the tangent line equation is

$$y - 10e = 12e(x - 5)$$

■ 3. A particle moves along a curve and its position is represented by $s(t) = t^3 + \cos(\pi t)$. Find the acceleration of the particle after 2 seconds.

A -13 units/sec²

B $-12 + \pi^2$ units/sec²

[✓ C] $12 - \pi^2$ units/sec²

D 11 units/sec²

Solution: C

Find the first and second derivative for $s(t)$, making sure to apply chain rule.

$$s(t) = t^3 + \cos(\pi t)$$

$$v(t) = s'(t) = 3t^2 - \pi \sin(\pi t)$$

$$a(t) = s''(t) = 6t - \pi^2 \cos(\pi t)$$

Substitute $t = 2$ into $a(t)$.

$$a(2) = 6(2) - \pi^2 \cos(\pi(2))$$

$$a(2) = 12 - \pi^2 \cos(2\pi)$$

$$a(2) = 12 - \pi^2(1)$$

$$a(2) = 12 - \pi^2$$

■ 4. Given that the velocity of a particle can be modeled by $v(t) = t^2 - 8t + 15$ for $t \geq 0$, on what time interval(s) is the particle speeding up?

A $(4,\infty)$

B $(0,4)$

C $(0,3) \cup (4,5)$

$\boxed{\text{D}}$ $(3,4) \cup (5,\infty)$

Solution: D

In order to determine whether the particle is speeding up, we need to compare the signs of velocity and acceleration. If velocity and acceleration have the same sign, the particle is speeding up.

$$v(t) = t^2 - 8t + 15$$

$$v(t) = (t - 5)(t - 3)$$

So $v(t) = 0$ when $t = 3, 5$. If we create a sign chart, we determine that

$v(t) > 0$ on $(0,3)$ and $(5,\infty)$

$v(t) < 0$ on $(3,5)$

Find acceleration by differentiating velocity.

$$a(t) = 2t - 8$$

$$a(t) = 2(t - 4)$$

So $a(t) = 0$ when $t = 4$. If we create a sign chart, we determine that

$a(t) < 0$ on $(0,4)$

$a(t) > 0$ on $(4,\infty)$

Therefore, the particle is speeding up on $(3,4)$ because velocity and acceleration are both negative on that interval. It's also speeding up on $(5,\infty)$ because velocity and acceleration are both positive on that interval.

■ 5. Given $g(x)$ is a twice differentiable function and that the tangent line to $g(x)$ at $x = 3$ was used to approximate the value of $g(2.9)$, which piece of information would guarantee that $g(2.9)$ is an overestimate of the true value?

$\boxed{\text{A}}$ $g(x)$ is concave down on the interval $(2.9,3)$

B $g(x)$ is concave up on the interval $(2.9,3)$

C $g(x)$ is increasing on the interval $(2.9,3)$

D $g(x)$ is decreasing on the interval $(2.9,3)$

Solution: **A**

The tangent line to a curve will lie above the function on any interval where the original function is concave down. If the function is concave down on $(2.9,3)$, then the tangent line will be above the curve and therefore have y-values larger than the actual y-values along the curve.

■ 6. If $y=3x^2-2x$ and $u=4x-1$, find $\frac{dy}{du}$.

[✓ A] $\frac{3x-1}{2}$

B $\frac{12x^2-6x+1}{(4x-1)^2}$

C $24x+1$

D $6x-2$

Solution: **A**

This is the chain rule in disguise. We're looking for $\frac{dy}{du}$, which we can find as

$$\frac{dy}{du}=\frac{dy}{dx}\cdot\frac{dx}{du}$$

If we differentiate $y=3x^2-2x$, we find

$$\frac{dy}{dx}=6x-2$$

Rearrange $u=4x-1$ to solve for x.

$$u+1=4x$$

$$x = \frac{u+1}{4}$$

If we differentiate this equation for x, we get

$$\frac{dx}{du} = \frac{1}{4}$$

Then $\frac{dy}{du}$ is

$$\frac{dy}{du} = (6x-2)\left(\frac{1}{4}\right)$$

$$\frac{dy}{du} = \frac{6x-2}{4}$$

$$\frac{dy}{du} = \frac{3x-1}{2}$$

■ 7. Air is being sucked out of a spherical balloon so that its volume is decreasing by 250 cm^3/s. How fast is the radius decreasing when the radius is 5 cm?

A $\frac{5}{2\pi}$ cm/s

[✓ B] $-\frac{5}{2\pi}$ cm/s

C $\frac{2}{5\pi}$ cm/s

D $-\frac{2}{5\pi}$ cm/s

Solution: B

The formula for the volume of a sphere is

$$V=\frac{4}{3}\pi r^3$$

Use implicit differentiation to take the derivative of both sides.

$$(1)\frac{dV}{dt}=\frac{4}{3}\pi 3r^2\frac{dr}{dt}$$

$$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$$

From the question, we know that $dV/dt=-250$ and that $r=5$, so we'll plug those in.

$$-250=4\pi(5)^2\frac{dr}{dt}$$

$$-250=100\pi\frac{dr}{dt}$$

Solve for dr/dt, which is the rate we were asked to find.

$$\frac{dr}{dt}=-\frac{250}{100\pi}$$

$$\frac{dr}{dt}=-\frac{5}{2\pi}$$

■ 8. A point moves along the curve $y=2x^2-1$ in such a way that y is decreasing at a rate of 2 units per second. At what rate is x changing when $x=\frac{3}{2}$?

A Decreasing at $\frac{7}{2}$ units/s

B Increasing at $\frac{7}{2}$ units/s

$\boxed{\text{C}}$ Decreasing at $\frac{1}{3}$ units/s D Increasing at $\frac{1}{3}$ units/s

Solution: C

Differentiate both sides of $y = 2x^2 - 1$ with respect to time t.

$$\frac{d}{dt}[y = 2x^2 - 1]$$

$$\frac{dy}{dt} = 4x\frac{dx}{dt}$$

We know from the problem that $\frac{dy}{dt} = -2$ when $x = \frac{3}{2}$, so we substitute those values.

$$-2 = 4\left(\frac{3}{2}\right)\frac{dx}{dt}$$

$$-2 = 6\frac{dx}{dt}$$

$$-\frac{2}{6} = \frac{dx}{dt}$$

$$\frac{dx}{dt} = -\frac{1}{3}$$

So x is decreasing at a rate of $\frac{1}{3}$ units/s.

■ 9. An item is currently selling for $\$150$/unit. The quantity supplied is decreasing by 25 units/week. At what rate is the price of the item changing, if $q = 4{,}000e^{-0.01p}$?

A $\$1.03$ per week

$\boxed{\text{B}}$ $\$2.80$ per week

C $\$2.64$ per week

D $\$0.97$ per week

Solution: B

Use implicit differentiation to take the derivative of both sides of the quantity equation.

$$q = 4{,}000e^{-0.01p}$$

$$\frac{dq}{dt} = -40e^{-0.01p}\ \frac{dp}{dt}$$

From the question, we know that $p = 150$ and $dq/dt = -25$, so we'll plug those in.

$$-25 = -40e^{-0.01(150)}\ \frac{dp}{dt}$$

$$-25 = -40e^{-1.50}\ \frac{dp}{dt}$$

Solve for dp/dt, which is the rate we were asked to find.

$$\frac{dp}{dt} = \frac{-25}{-40e^{-1.50}}$$

$$\frac{dp}{dt} = \frac{5}{8e^{-1.50}}$$

$$\frac{dp}{dt}=\frac{5e^{1.50}}{8}$$

$$\frac{dp}{dt}\approx \$2.80$$

■ **10. Evaluate the limit $\lim_{x\to\infty}\frac{2x-e^x}{4x+\ln x}$.**

A ∞

B $\frac{1}{2}$

[✓ C] $-\infty$

D 0

Solution: C

Because the limit of the numerator and denominator both go to ∞ as $x\to\infty$, we can apply L'Hospital's rule. The derivative of the numerator and denominator are

$$\frac{d}{dx}(2x-e^x)=2-e^x$$

$$\frac{d}{dx}(4x+\ln x)=4+\frac{1}{x}$$

So rewrite the limit as

$$\lim_{x\to\infty}\frac{2-e^x}{4+\frac{1}{x}}$$

Then when we evaluate the limit, the numerator approaches $-\infty$ and the denominator approaches 4, so the limit will be $-\infty$.

■ 11. Sand falling from a chute forms a conical pile whose height is always equal to $\frac{4}{3}$ of the radius of the base. How fast is the volume changing when the radius of the base is 6 inches and increasing at a rate of $\frac{1}{2}$ inches per minute? The volume of a cone is given by $V=\frac{1}{3}\pi r^2h$.

Solution:

We know that $h=\frac{4}{3}r$.

$$V=\frac{1}{3}\pi r^2h$$

$$V=\frac{1}{3}\pi r^2\left(\frac{4}{3}r\right)$$

$$V=\frac{4}{9}\pi r^3$$

Differentiate both sides with respect to time t.

$$\frac{dV}{dt}=\frac{4}{9}\pi(3r^2)\frac{dr}{dt}$$

$$\frac{dV}{dt}=\frac{4}{3}\pi r^2\frac{dr}{dt}$$

Substitute $r=6$ and $\frac{dr}{dt}=\frac{1}{2}$.

$$\frac{dV}{dt} = \frac{4}{3}\pi(6)^2\left(\frac{1}{2}\right)$$

$$\frac{dV}{dt} = \frac{4}{6}\pi(36)$$

$$\frac{dV}{dt} = 24\pi \text{ in}^3\text{/min}$$

■ 12. The position of a particle moving along the x-axis is represented by $x(t) = \sqrt[3]{t^3 + 2t}$, where $x(t)$ represents the position at t seconds on the interval $0 \leq t \leq 10$.

a. Determine the average velocity of the particle on $[0,2]$ (from 0 seconds to 2 seconds).

b. Determine the velocity function.

c. Determine the acceleration of the particle when $t = 2$ seconds.

Solution:

a. Average velocity is change in position over change in time.

$$v_{avg} = \frac{x(2) - x(0)}{2 - 0}$$

$$v_{avg} = \frac{\sqrt[3]{2^3 + 2(2)} - \sqrt[3]{0^3 + 2(0)}}{2 - 0}$$

$$v_{avg} = \frac{\sqrt[3]{8 + 4}}{2}$$

$$v_{avg} = \frac{\sqrt[3]{12}}{2}$$

$$v_{avg} \approx 1.1447$$

b. Finding the velocity function requires chain rule.

$$x(t) = \sqrt[3]{t^3 + 2t}$$

$$x'(t) = \frac{1}{3}(t^3 + 2t)^{-\frac{2}{3}}\frac{d}{dt}(t^3 + 2t)$$

$$x'(t) = \frac{1}{3(t^3 + 2t)^{\frac{2}{3}}}(3t^2 + 2)$$

$$x'(t) = \frac{3t^2 + 2}{3(t^3 + 2t)^{\frac{2}{3}}}$$

c. The first derivative is

$$x'(t) = \frac{3t^2 + 2}{3(t^3 + 2t)^{\frac{2}{3}}}$$

and the second derivative is

$$x''(t) = \frac{(6t)(3(t^3 + 2t)^{\frac{2}{3}}) - (3t^2 + 2)(2(t^3 + 2t)^{-\frac{1}{3}})}{(3(t^3 + 2t)^{\frac{2}{3}})^2}$$

$$x''(t) = \frac{18t(t^3 + 2t)^{\frac{2}{3}} - 2(3t^2 + 2)(t^3 + 2t)^{-\frac{1}{3}}}{9(t^3 + 2t)^{\frac{4}{3}}}$$

At $t = 2$, the value of the second derivative is

$$x''(2) = \frac{18(2)(2^3 + 2(2))^{\frac{2}{3}} - 2(3(2)^2 + 2)(2^3 + 2(2))^{-\frac{1}{3}}}{9(2^3 + 2(2))^{\frac{4}{3}}}$$

$$x''(2) = \frac{36(8+4)^{\frac{2}{3}} - 2(3(4)+2)(8+4)^{-\frac{1}{3}}}{9(8+4)^{\frac{4}{3}}}$$

$$x''(2) = \frac{36(12)^{\frac{2}{3}} - 2(12+2)(12)^{-\frac{1}{3}}}{9(12)^{\frac{4}{3}}}$$

$$x''(2) = \frac{36(12)^{\frac{2}{3}} - 28(12)^{-\frac{1}{3}}}{9(12)^{\frac{4}{3}}}$$

$$x''(2) \approx 0.7137$$

■ 13. Water is entering a tank at the rate $E(t) = 9 - 2^{0.3t}$ gallons/minute and leaving at a rate of $L(t) = 1 + x\tan^{-1}x$ gallons/minute.

a. Find $E'(2)$. Explain the meaning of $E'(2)$ in the context of the problem.

b. Is the level of water in the tank increasing or decreasing at $t = 5$? Justify your answer.

c. Assuming there were 18 gallons of water in the tank at $t = 5$, use a linear approximation to estimate the amount of water in the tank at $t = 5.25$ minutes.

Solution:

a. $E'(2) = -0.315$. This means water is entering the tank at a rate that's decreasing by 0.315 gallons/minute.

b. $E(5) = 6.172$ and $L(5) = 7.867$ so the water level in the tank is decreasing because more water is leaving than entering each minute.

c. The rate of change for the volume of the water in the tank is $6.172 - 7.867 = -1.695$, which is the slope of the function modeling the amount of water in the tank. We also know from the problem that $(5,18)$ satisfies this function. So the tangent line would be $y - 18 = -1.695(x - 5)$, and if we substitute $x = 5.25$, we get approximately 17.576 gallons of water in the tank at time $t = 5.25$.