4.1 INTERPRETING THE MEANING OF THE DERIVATIVE IN CONTEXT

1. A population is modeled by $p(t) = 1.21(1.0144)^t$, where t is the number of years since 2010. Match the notation on the left with the corresponding statement on the right.

$p(5)$	The number of people by which the population is changing in the year 2015
$p'(5)$	The rate at which the number of people in the population is changing in the year 2015
$p''(5)$	The total population in 2015

2. Every day, Kaylin records her weight. She models the data with $w(t) = -2.5t^4 - 0.1x^2 + 0.2x + 146$, where t is the number of days since she started recording her weight. Explain the meaning of each statement.

   a. $w'(7) < 0$: Kaylin is losing weight on day 7.

   b. $w'(7) > 0$: Kaylin is gaining weight on day 7.

   c. $w'(t) < 0$ for $t < 7$, $w'(t) > 0$ for $t > 7$ and $w'(7) = 0$:

   Kaylin reaches a relative minimum weight on day 7.

3. A drone moves along the curve $h(t) = \dfrac{1}{3}t^3 - 2t^2 + 3t$, where $h(t)$ gives the number of feet traveled after t seconds. Complete the table.

t	0	3
h(t)	0	0
h'(t)	3	0
h''(t)	-4	2

4.2 STRAIGHT-LINE MOTION: CONNECTING POSITION, VELOCITY, AND ACCELERATION

1. Velocity is the first derivative of ____position____, and acceleration is its second derivative. So acceleration is the first derivative of ____velocity____. ____Speed____ is the absolute value of velocity.

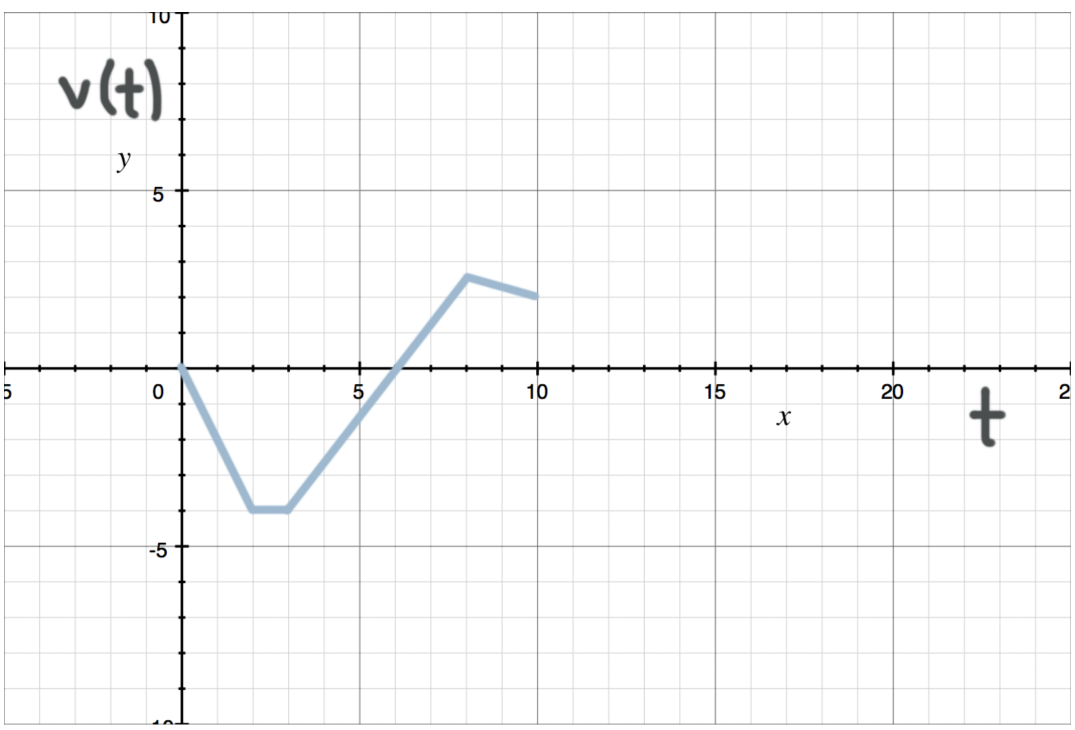
2. Hannah rides a bike from left to right along a straight road. Say whether each statement is true or false, and explain why.

If Hannah is moving to the left, her velocity is positive.	False. Movement to the left is associated with negative velocity.
If Hannah is moving to the left and her velocity is decreasing, Hannah is speeding up.	True. She's speeding up because the signs of velocity and acceleration are the same. Her velocity is negative because she's moving to the left, and acceleration is negative because velocity is decreasing.
If Hannah's velocity changes from negative to positive or from positive to negative, it means she's changed direction.	True. This indicates a shift in her position from left to right, or from right to left, meaning that Hannah is changing direction.
If Hannah biked 20 miles and finished her ride where she started, her displacement is 20 miles.	False. Her distance is 20 miles, but distance and displacement are not the same thing. Displacement is the difference between her starting and ending positions, so her displacement is 0.

3. A train's position on an east-west track is modeled by $x(t) = t^3 - 3t^2 - 10$, where time t is measured in minutes and position $x(t)$ is measured in miles. Complete the table with the units of each value in miles, miles/minute, or miles/minute^2.

Position	miles
Displacement	miles
Total distance	miles
Velocity	miles/minute
Acceleration	miles/minute^2
Average velocity	miles/minute
Average acceleration	miles/minute^2
Speed	miles/minute

4. The graph shows velocity $v(t)$, measured in feet per second, of a squirrel running across a telephone line. Say in the table where velocity and acceleration are positive, negative, or zero, and whether the squirrel is speeding up, slowing down, or neither.



Time (seconds)	Velocity (ft/sec)	Acceleration (ft/sec^2)	Speeding up, slowing down, or neither
(0,2)	Negative	Negative	Speeding up
(2,3)	Negative	Zero	Neither
(3,6)	Negative	Positive	Slowing down
(6,8)	Positive	Positive	Speeding up
(8,10)	Positive	Negative	Slowing down

5. The table shows information about a marble that's rolling back and forth, where position is given by $x(t)$ after time t seconds.

t	0	3	6	9
x(t)	0	2	1	5
v(t)	1.5	-1	3	4
a(t)	-2	4	2	4

a. Find displacement from 0 seconds to 9 seconds.

$x(9) - x(0) = 5$ inches

b. Find average velocity from 3 seconds to 9 seconds.

$$\frac{x(9) - x(3)}{9 - 3} = \frac{5 - 2}{6} = \frac{3}{6} = \frac{1}{2} \text{ feet/second}$$

c. Find average acceleration from 0 seconds to 6 seconds.

$$\frac{v(6) - v(0)}{6 - 0} = \frac{3 - 1.5}{6} = \frac{1.5}{6} = \frac{1}{4} \text{ feet/second}^2$$

4.3 RATES OF CHANGE IN APPLIED CONTEXT OTHER THAN MOTION

1. Given the area of a circle $A = \pi r^2$, whose radius is r,

   a. Find rate of change of the area with respect to the radius.

      Differentiate A with respect to r to get $\frac{dA}{dr} = 2\pi r$

   b. How fast is the circle's area changing when $r = 5$ cm?

      $\frac{dA}{dr} = 2\pi r = 2\pi(5) = 10\pi$ cm^2

   c. Interpret the answer to b.

      At the moment when the radius is 5 cm long, the area of the circle is growing by 10π cm^2.

2. The cost of producing m cellphones is $c(m) = 300 + 100m - 0.2m^2$.

   a. Find the average cost of producing 100 cellphones and explain what the answer represents.

      $\frac{c(100) - c(0)}{100 - 0} = \frac{8{,}300 - 300}{100} \approx 27.67$, which says that the average cost of producing 100 cellphones is $\$27.67$ per phone.

   b. Find instantaneous rate of change when 50 cellphones are produced. Explain what the answer represents.

      $c'(m) = 100 - 0.4m$ and $c'(50) = 100 - 0.4(50) = 80$, so the instantaneous rate of change in cost per cellphone at $m = 50$ cellphones is $\$80$ per cellphone.

4.4 INTRODUCTION TO RELATED RATES

1. A circle's radius and area are related by $A = \pi r^2$. If the circle is growing, write an equation that relates the rate of change of the area A with respect to time t, to the rate of change of the radius r with respect to time t.

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

2. Mulch is being dropped from a dump truck at a rate of 20 cubic feet per minute, and it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 5 feet high? In the context of this problem, match each value to a description.

   a. Height and base of the cone. [Quantities that vary depending on time]

   b. Rate at which mulch is being dumped. [Quantity that's not changing]

   c. Rate at which the pile is growing when the height of the pile is 5 feet. [Quantity that exists at only a specific moment in time]

3. A 10-foot ladder is leaning against a house when the base starts to slide away. When the base is 8 feet from the house, it's sliding away at 3 ft/sec. Fill in the blanks with "constant," "positive," or "negative."

   $\frac{dx}{dt}$, the distance along the ground between the house and the base of the ladder, is _____positive________,

$\frac{dy}{dt}$, the distance along the wall between the top of the ladder and the ground, is ____negative______, and

$\frac{dz}{dt}$, the length of the ladder, is _____constant______.

4.5 SOLVING RELATED RATES PROBLEMS

1. A 10-foot ladder is leaning against a house when the base starts to slide away. When the base is 8 feet from the house, it's sliding away at 3 ft/sec. How fast is the ladder sliding down the wall at this moment?

Sketch a right triangle with a as the leg along the ground, b as the leg up the wall, and c as the ladder forming the hypotenuse.

When the base is $a = 8$ feet from the house and the length of the ladder is $c = 10$,

$$8^2 + b^2 = 10^2$$

$$64 + b^2 = 100$$

$$b^2 = 36$$

$$b = 6$$

Then solve for the rate at which the ladder is sliding down the wall.

$$a^2 + b^2 = c^2$$

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

$$2(8)(3) + 2(6)\frac{db}{dt} = 2(8)(0)$$

$$48 + 12\frac{db}{dt} = 0$$

$$12\frac{db}{dt} = -48$$

$$\frac{db}{dt} = -4$$

The ladder is sliding down the wall at 4 feet/second.

2. A spherical balloon is inflated at a rate of 25 cm^3/sec. How fast is the balloon's radius increasing when the radius is 5 cm. The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$.

$$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

$$25 = 4\pi(5)^2\frac{dr}{dt}$$

$$25 = 4\pi(25)\frac{dr}{dt}$$

$$1 = 4\pi\frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{4\pi}$$

3. Let $y = f(x) = 4x^2 - 2x + 1$. If $\frac{dx}{dt} = -3$ cm/sec, complete the table.

x	0	1	2
dy/dt	6	-18	-42

4.6 APPROXIMATING VALUES OF A FUNCTION USING LOCAL LINEARITY AND LINEARIZATION

1. For $f(x) = x^3 - 4$, complete the table.

x	-2	0	1	3
f(x)	-12	-4	-3	23
f'(x)	12	0	3	27
Equation of the tangent line	y+12=12(x+2)	y=-4	y+3=3(x-1)	y-23=27(x-3)

2. Write an equation for the line tangent to $f(x) = x^{\frac{1}{3}}$ at $x = 8$, and use it to approximate $f(8.1)$.

Find the value of the derivative at $x = 8$.

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$$

$$f'(8) = \frac{1}{3}(8)^{-\frac{2}{3}}$$

$$f'(8) = \frac{1}{3(8)^{\frac{2}{3}}}$$

$$f'(8) = \frac{1}{3(2)^2}$$

$$f'(8) = \frac{1}{12}$$

Find the value of the function at $x = 8$.

$$f(8) = 8^{\frac{1}{3}}$$

$$f(8) = 2$$

Then the tangent line equation is

$$y - 2 = \frac{1}{12}(x - 8)$$

$$y = \frac{1}{12}(x - 8) + 2$$

And an approximation of $f(8.1)$ is

$$f(8.1) \approx \frac{1}{12}(8.1 - 8) + 2$$

$$f(8.1) \approx 2.008$$

4.7 USING L'HOSPITAL'S RULE FOR DETERMINING LIMITS OF INDETERMINATE FORMS

1. Circle the limits that give indeterminate forms.

$$\lim_{x\to 2} \frac{x-2}{x+2} \qquad \lim_{x\to 2} \frac{2-\sqrt{x^3-6}}{x-2} \qquad \lim_{x\to 2} \frac{x+2}{x-2} \qquad \lim_{x\to 0} \frac{e^x - \cos x}{x} \qquad \lim_{x\to\infty} \frac{e^x}{\ln x}$$

2. For continuous functions $f(x)$ and $g(x)$, with $\lim_{x\to 1} f(x) = 0$ and $\lim_{x\to 1} g(x) = 0$,

a. True or False? $\lim_{x\to 1} \frac{f(x)}{g(x)} = \frac{0}{0}$.

False. $0/0$ is a form, not a number. It's inaccurate to say that something can equal $0/0$. Remember to always split up the limit of the numerator and denominator when stating that they equal 0.

b. True or False? $\lim_{x\to 1} \frac{f(x)}{g(x)} = \lim_{x\to 1} \frac{g(x)f'(x) - f(x)g(x)}{[g(x)]^2}$

False. We don't take the derivative of the quotient when applying L'Hospital's Rule. Instead, we take the derivative of the numerator and denominator separately.

c. True or False? $\lim_{x\to 1} \frac{f(x)}{g(x)} = \lim_{x\to 1} \frac{f'(x)}{g'(x)}$

True. This is L'Hospital's Rule.

d. True or False? $\lim_{x\to 2} \frac{f(x)}{g(x)} = \frac{f'(2)}{g'(2)}$

False. We can't know this for sure without knowing whether or not $f(x)$ and $g(x)$ are differentiable at $x = 2$. We could make this statement definitely true by changing it to $\lim_{x\to 2} \frac{f(x)}{g(x)} = \lim_{x\to 2} \frac{f'(x)}{g'(x)}$.

3. Let $\lim_{x\to0}\frac{\sin x}{x}=\lim_{x\to0}\frac{f(x)}{g(x)}$. Complete the table and fill in the statement.

	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
$\lim_{x\to0}$	0	0	1	1

Because

$$\lim_{x\to0}\sin x=0 \text{ and } \lim_{x\to0}x=0,$$

the given limit is indeterminate. Applying L'Hospital's Rule gives

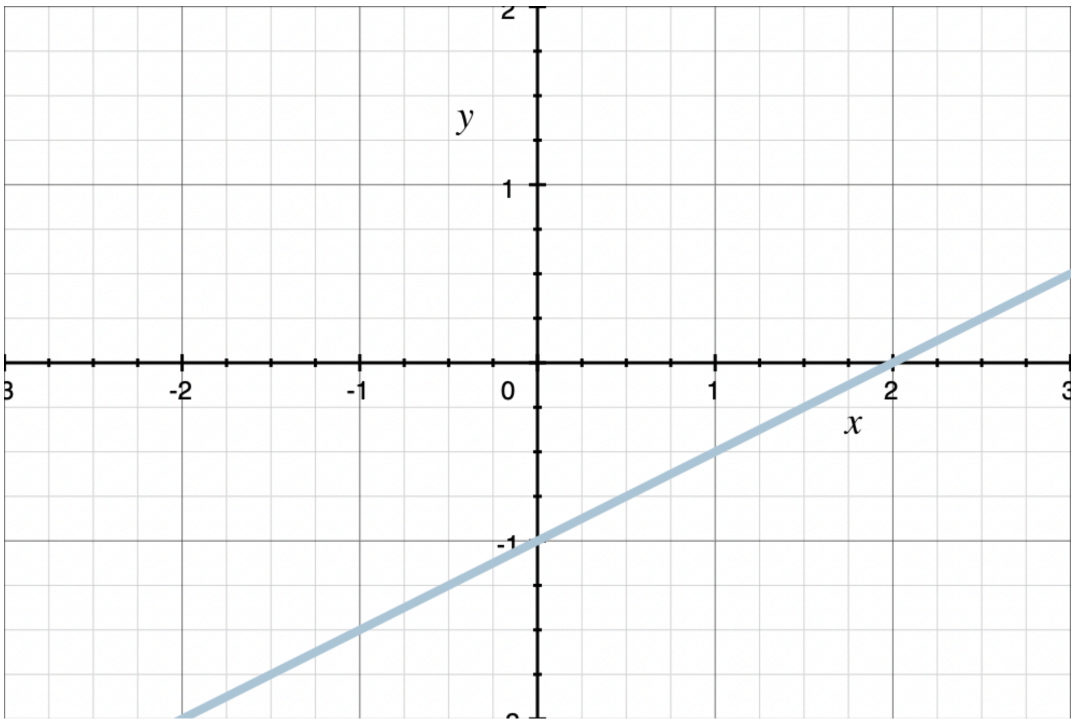
$$\lim_{x\to0}\frac{\sin x}{x}=\lim_{x\to0}\frac{f'(x)}{g'(x)}=\lim_{x\to0}\frac{\cos x}{1}=1.$$

4. Because $\lim_{x\to0}(e^x-\cos x)=0$ and $\lim_{x\to0}x=0$, then L'Hospital's rule applies to $\lim_{x\to0}\frac{e^x-\cos x}{x}$, and the value of the limit is

$$\lim_{x\to0}\frac{e^x-\cos x}{x}=\lim_{x\to0}\frac{e^x+\sin x}{1}=1.$$

5. Given the table of values for $f(x)$ and the graph of $g(x)$, for each limit, say whether or not L'Hospital's Rule applies, and find the limit.

x	0	2	4
f(x)	3	0	5
f'(x)	-2	4	0



a. $\lim_{x\to 2}\frac{f(x)}{g(x)}$

The limit gives an indeterminate form, so L'Hospital's Rule applies, and $\lim_{x\to 2}\frac{f'(x)}{g'(x)}=\frac{\lim_{x\to 2}f'(x)}{\lim_{x\to 2}g'(x)}=\frac{f'(2)}{g'(2)}=\frac{4}{\frac{1}{2}}=8$.

b. $\lim_{x\to 0}\frac{f(x)}{g(x)}$

The limit does not give an indeterminate form, so L'Hospital's Rule does not apply, and $\lim_{x\to 0}\frac{f(x)}{g(x)}=\frac{\lim_{x\to 0}f(x)}{\lim_{x\to 0}g(x)}=\frac{3}{-1}=-3$.

6. Evaluate $\lim_{x\to 0}\frac{\sin^2 x}{x^2}$.

Using substitution gives an indeterminate form, so apply L'Hospital's Rule.

$$\lim_{x\to 0}\frac{2\sin x\cos x}{2x}$$

$$\lim_{x\to 0}\frac{\sin x\cos x}{x}$$

Substitution again gives an indeterminate form, so apply L'Hospital's Rule a second time.

$$\lim_{x\to 0}\frac{(\cos x)(\cos x)+(\sin x)(-\sin x)}{1}$$

$$\lim_{x\to 0}\cos^2 x-\sin^2 x$$

Now substitution gives a real-number answer for the limit.

$$\cos^2(0)-\sin^2(0)$$

$$1^2-0^2$$

$$1$$