■ 1. Determine whether or not the Mean Value Theorem applies to $f(x)=-\frac{1}{x}$ on the interval $\left[-3,-\frac{1}{2}\right]$. If it applies, find any value(s) of c in the interval such that $f'(c)$ is equivalent to the average rate of change over the interval. If the MVT does not apply, say why.

$\boxed{\text{A}}$ The MVT applies, and $c=-\sqrt{\frac{3}{2}}$

B The MVT does not apply because f is not continuous at $x=0$

C The MVT does not apply because $f\left(-\frac{1}{2}\right)\neq f(-3)$

D The MVT does not apply because $f'(c)$ does not exist

Solution: A

The Mean Value Theorem does apply, because the function is both continuous and differentiable on the given interval. The function is discontinuous at $x=0$, but that's outside the interval. The average rate of change on the interval is

$$\frac{f(x)-f(a)}{x-a}=\frac{f\left(-\frac{1}{2}\right)-f(-3)}{-\frac{1}{2}-(-3)}=\frac{2-\frac{1}{3}}{-\frac{1}{2}+3}=\frac{\frac{5}{3}}{\frac{5}{2}}=\frac{2}{3}$$

Find the derivative,

$$f(x)=-\frac{1}{x}=-x^{-1}$$

$$f'(x)=x^{-2}=\frac{1}{x^2}$$

then evaluate it at c to get $f'(c)$, one side of the MVT equation.

$$f'(c) = \frac{1}{c^2}$$

Set up the MVT equation and solve it for c.

$$f'(c) = \frac{f(x) - f(a)}{x - a}$$

$$\frac{1}{c^2} = \frac{2}{3}$$

$$3 = 2c^2$$

$$c^2 = \frac{3}{2}$$

$$c = \pm\sqrt{\frac{3}{2}}$$

Only $c = -\sqrt{\frac{3}{2}}$ falls within the given interval (the positive value is outside the interval), so $c = -\sqrt{\frac{3}{2}}$.

■ 2. Find all critical numbers of $f(x) = (9 - x^2)^{\frac{3}{5}}$.

A 0

B 3

C $-3, 3$

[✓ D] $-3, 0, 3$

Solution: D

Find the derivative.

$$f'(x) = \frac{3}{5}(9 - x^2)^{-\frac{2}{5}}(-2x)$$

$$f'(x) = -\frac{6}{5}x(9 - x^2)^{-\frac{2}{5}}$$

$$f'(x) = -\frac{6x}{5(9 - x^2)^{\frac{2}{5}}}$$

The derivative is 0 when

$$6x = 0$$

$$x = 0$$

The derivative is undefined when

$$5(9 - x^2)^{\frac{2}{5}} = 0$$

$$9 - x^2 = 0$$

$$x^2 = 9$$

$$x = \pm 3$$

Then the critical numbers of the function are $x = -3, 0, 3$.

■ 3. Given $f'(x) = x^2 - 8x - 9$, on what intervals is $f(x)$ increasing?

A $(-1,9)$

B $(-9,1)$

C $(-\infty, -9) \cup (1,\infty)$

$\boxed{\text{D}}$ $(-\infty, -1) \cup (9,\infty)$

Solution: D

Set the derivative equal to 0 and solve for x.

$$x^2-8x-9=0$$

$$(x-9)(x+1)=0$$

$$x=-1,\ 9$$

Evaluate the derivative at a test value in each interval.

$$f'(-2)=(-2)^2-8(-2)-9=11>0$$

$$f'(0)=0^2-8(0)-9=-9<0$$

$$f'(10)=10^2-8(10)-9=11>0$$

Because the derivative is positive on $(-\infty,-1)\cup(9,\infty)$, the function $f(x)$ is increasing on $(-\infty,-1)\cup(9,\infty)$.

■ 4. Find the absolute minimum of $f(x)=\dfrac{10}{x^2+1}$ on the interval $[-1,2]$.

[✓ A] $(2,2)$

B $(0,10)$

C $(-1,5)$

D $\left(-\dfrac{1}{2},8\right)$

Solution: A

Find the derivative, $f'(x)$.

$$f'(x) = \frac{0(x^2+1) - 10(2x)}{(x^2+1)^2}$$

$$f'(x) = \frac{-20x}{(x^2+1)^2}$$

The derivative will equal 0 when the numerator is equal to 0.

$$-20x = 0$$

$$x = 0$$

Use the candidates test and evaluate the original function at the critical point, and both endpoints of the interval.

$$f(-1) = \frac{10}{(-1)^2+1} = \frac{10}{1+1} = \frac{10}{2} = 5$$

$$f(0) = \frac{10}{0^2+1} = \frac{10}{1} = 10$$

$$f(2) = \frac{10}{2^2+1} = \frac{10}{4+1} = \frac{10}{5} = 2$$

Based on these values, on the interval $[-1,2]$, the absolute minimum is at $x = 2$, specifically at $(2,2)$.

■ 5. At which value(s) of x does $f(x)$ have an inflection point, if $f''(x) = x^2(x-3)(x-6)^3$?

A $x = 0$

B $x = 0,\ 3,\ 6$

[✓ C] $x = 3,\ 6$

D $x = 0,\ 3$

Solution: C

To find inflection points of $f(x)$, set $f''(x) = 0$.

$$x^2(x-3)(x-6)^3 = 0$$

$$x = 0,\ 3,\ 6$$

These are potential inflection points, but we need to ensure that the second derivative changes sign at each one. We'll check the test points $x = -1,\ 1,\ 4,\ 7$.

$$f''(-1) = (-1)^2(-1-3)(-1-6)^3 = (-4)(-7)^3 = 1{,}372$$

$$f''(1) = 1^2(1-3)(1-6)^3 = (-2)(-5)^3 = (-2)(-125) = 250$$

$$f''(4) = 4^2(4-3)(4-6)^3 = 16(1)(-2)^3 = -128$$

$$f''(7) = 7^2(7-3)(7-6)^3 = 49(4)(1)^3 = 196$$

Therefore, we can say:

The second derivative is positive on $(-\infty,0)$

The second derivative is positive on $(0,3)$

The second derivative is negative on $(3,6)$

The second derivative is positive on $(6,\infty)$

Because the second derivative changes sign at $x = 3$ and $x = 6$, these are the inflection points of $f(x)$.

■ 6. A function $f(x)$ has critical points at $x = 0$ and $x = 2$. Use the Second Derivative Test to classify the critical points, if $f''(x) = -6x + 6$.

$\boxed{\text{A}}$ Relative minimum at $x = 0$; Relative maximum at $x = 2$

B Relative minimum at $x = 2$; Relative maximum at $x = 0$

C Relative minima at $x = 0$ and $x = 2$

D Relative maxima at $x = 0$ and $x = 2$

Solution: A

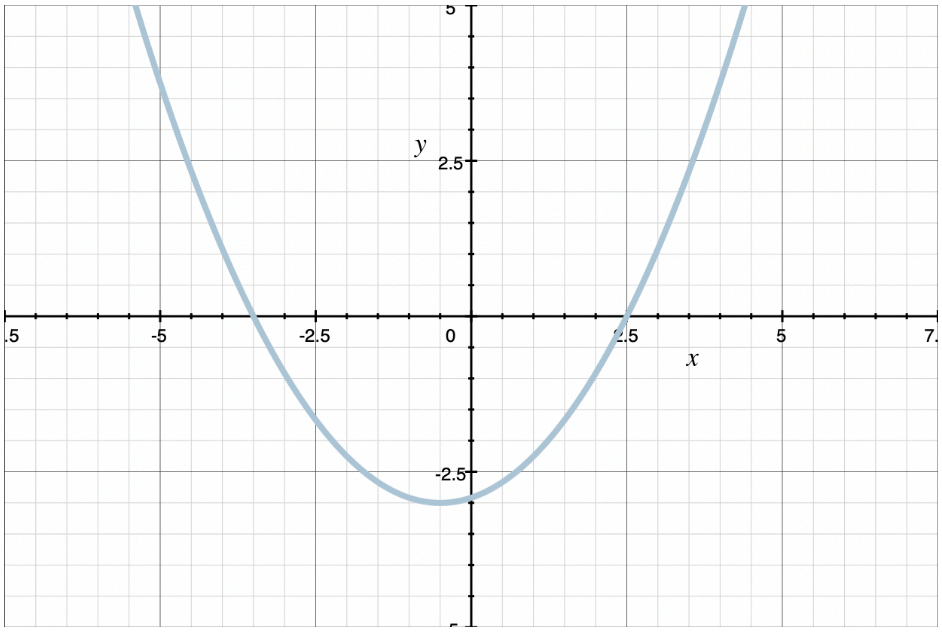
Evaluate the second derivative at both critical points.

$$f''(0) = -6(0) + 6 = 6 > 0$$

$$f''(2) = -6(2) + 6 = -6 < 0$$

Because the second derivative is positive at $x = 0$, the function $f(x)$ has a relative minimum there. Because the second derivative is negative at $x = 2$, the function so the function $f(x)$ as a relative maximum there.

■ 7. The graph of $f'(x)$ is shown below. Which of the following is not true about $f(x)$.



A $f(x)$ has a critical point at $x=-\frac{7}{2}$

[✓ B] $f(x)$ is increasing on $\left(-\frac{1}{2},\infty\right)$

C $f(x)$ has an inflection point at $x=-\frac{1}{2}$

D $f(x)$ is decreasing on $\left(-\frac{7}{2},\frac{5}{2}\right)$

Solution: B

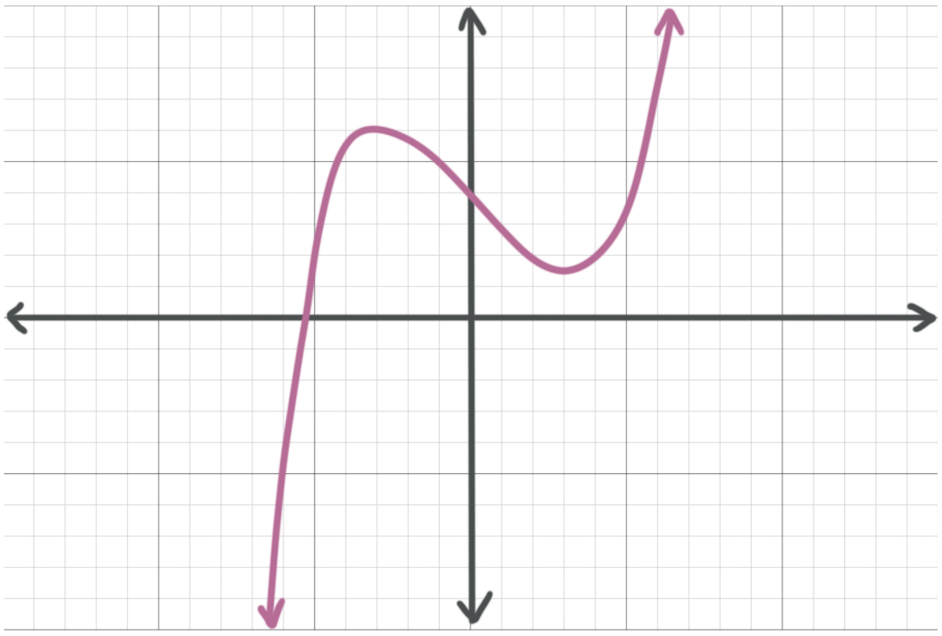
Answer choice A is true. The graph of $f'(x)$ has an x-intercept $x=-\frac{7}{2}$, so $f(x)$ has a critical point there.

Answer choice B is false. The graph of $f'(x)$ is negative on $\left(-\frac{1}{2},\frac{5}{2}\right)$, so $f(x)$ is decreasing on that interval, which means it isn't increasing on the entire interval $\left(-\frac{1}{2},\infty\right)$.

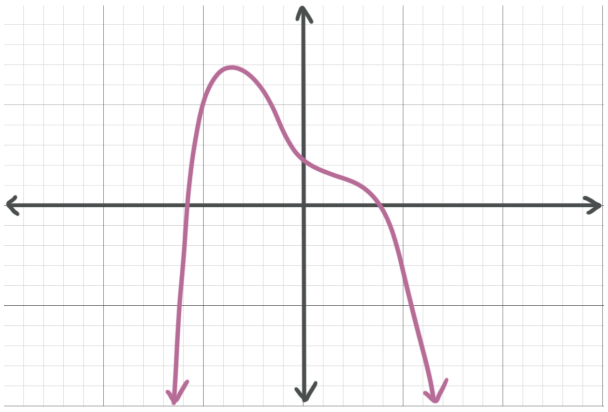
Answer choice C is true. The graph of $f'(x)$ has a critical point at $x = -\frac{1}{2}$, so $f(x)$ has an inflection point there.

Answer choice D is true. The graph of $f'(x)$ is negative on $\left(-\frac{7}{2}, \frac{5}{2}\right)$, so $f(x)$ is decreasing on that interval.

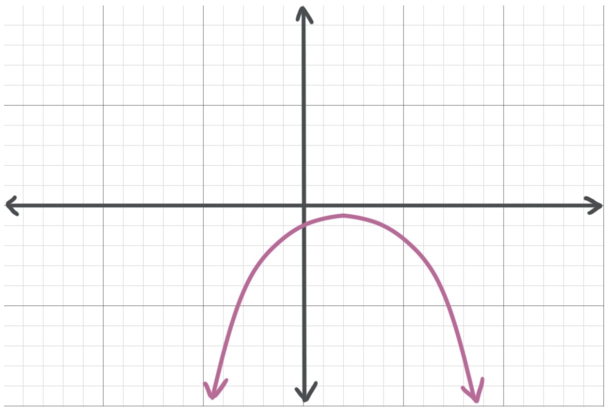
■ 8. Given the graph of $f'(x)$, which is a possible graph of $f(x)$?



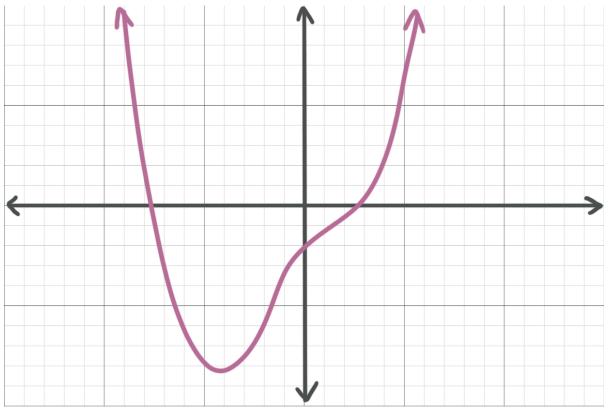
A



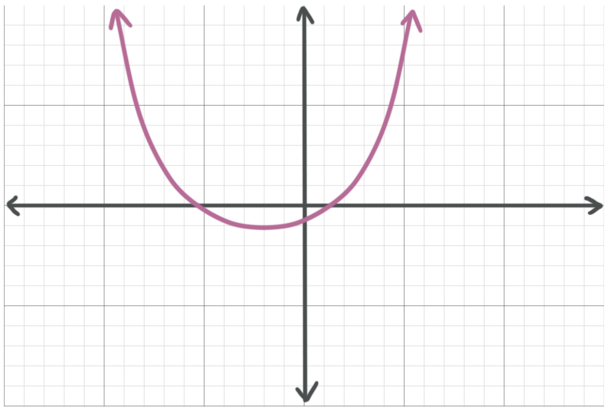
B



C



D

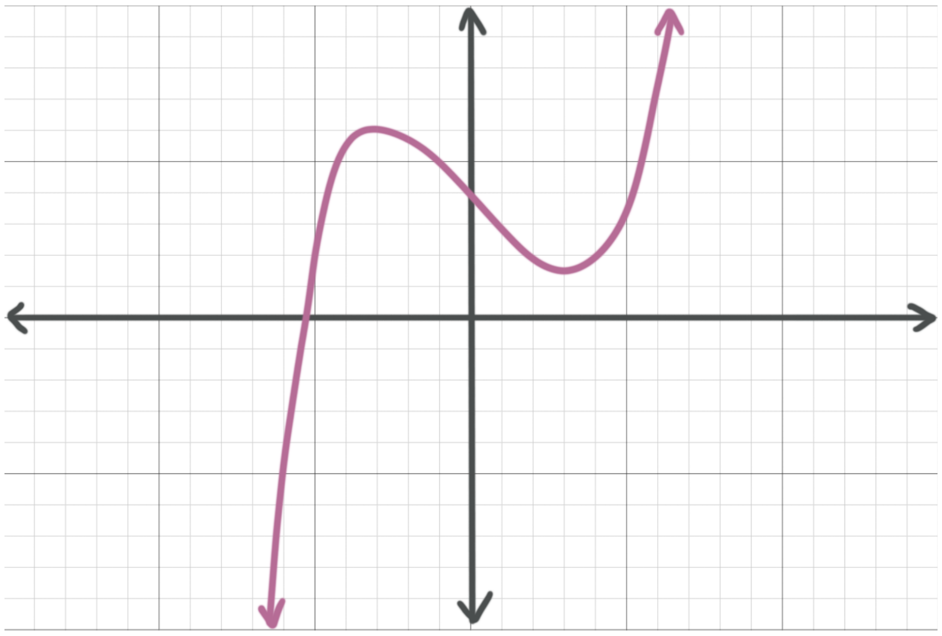


Solution: C

To take a sketch of $f'(x)$ and translate it into a sketch of a possible $f(x)$, we'll use the chart that compares the graphs of those two functions.

$f(x)$	$f'(x)$
Critical point	0
Increasing	**Positive**
Decreasing	**Negative**
Inflection point	**Critical point**
Concave up	**Increasing**
Concave down	**Decreasing**

If we consider the graph of $f'(x)$ we've been given,



we can see right away that, as we move from left to right, it's increasing, then decreasing, then increasing again. Which means that the graph of $f(x)$, as we move from left to right, must be concave up, then concave down, then concave up again.

The only graph matching that concavity pattern is the graph in answer choice C.

■ 9. How many horizontal tangents does the curve $y^2+3=x^2-e^y$ have?

[✓ A] One
B Two
C Three
D None

Solution: A

Find the derivative with respect to x.

$$\frac{d}{dx}(y^2+3=x^2-e^y)$$

$$2yy' = 2x - e^y y'$$

Solve for y'.

$$2yy' + e^y y' = 2x$$

$$y'(2y + e^y) = 2x$$

$$y' = \frac{2x}{2y + e^y}$$

Because $y' = 0$ when

$$2x = 0$$

$$x = 0$$

the curve has a critical point, and therefore a relative extremum, and also a horizontal tangent line, at $x = 0$. There is only one horizontal tangent line.

■ 10. Find the value(s) of c that satisfy the Mean Value Theorem for the function $h(x) = -\sqrt{25 - 5x}$ on the interval $[0,5]$.

Solution:

First, $h(x)$ is continuous and differentiable on the interval $[0,5]$, so the Mean Value Theorem applies. The problem says to find c in the interval such that

$$h'(c) = \frac{h(5) - h(0)}{5 - 0}$$

Find the values needed for the numerator.

$$h(5) = -\sqrt{25 - 5(5)} = -\sqrt{0} = 0$$

$$h(0) = -\sqrt{25 - 5(0)} = -\sqrt{25} = -5$$

Then

$$\frac{h(5) - h(0)}{5 - 0} = \frac{0 - (-5)}{5} = 1$$

Take the derivative,

$$h'(x) = -\frac{-5}{2\sqrt{25 - 5x}} = \frac{5}{2\sqrt{25 - 5x}}$$

then set $h'(x) = 1$ and solve for x.

$$\frac{5}{2\sqrt{25 - 5x}} = 1$$

$$\frac{5}{2} = \sqrt{25 - 5x}$$

$$\frac{25}{4} = 25 - 5x$$

$$5x = 25 - \frac{25}{4}$$

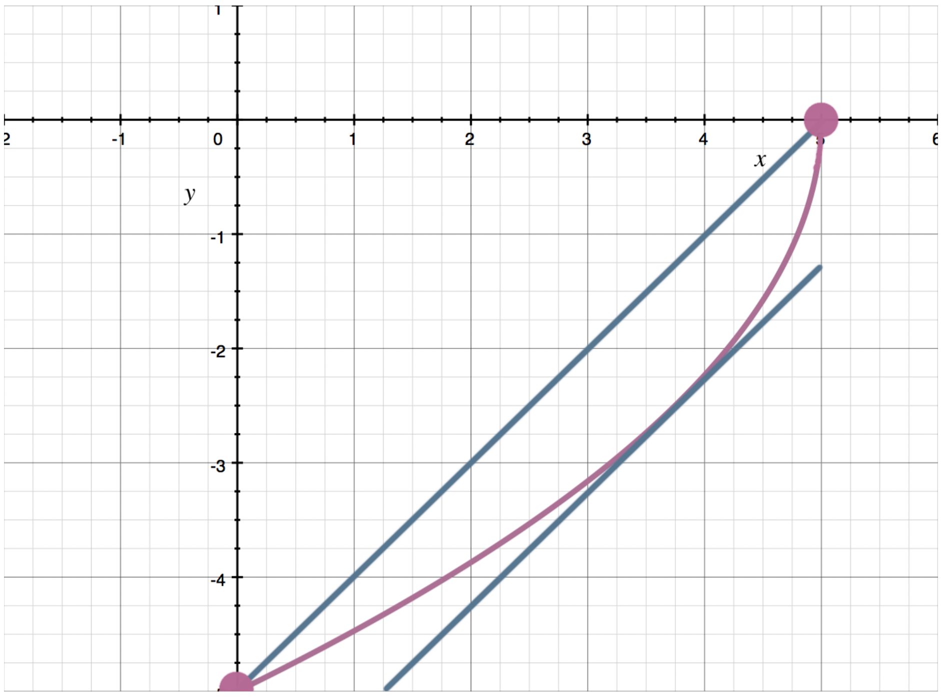
$$x = \frac{75}{20}$$

$$x = \frac{15}{4}$$

Verify that the slope of the tangent line at this value is 1.

$$h'\left(\frac{15}{4}\right)=\frac{5}{2\sqrt{25-5\left(\frac{15}{4}\right)}}=\frac{5}{2\sqrt{\frac{25}{4}}}=\frac{5}{2\left(\frac{5}{2}\right)}=\frac{5}{5}=1$$

Therefore, $c=\frac{15}{4}$. The figure illustrates how this point satisfies the Mean Value Theorem.



■ 11. For each question below about the function $f(x)=\frac{x^2-2x+4}{x-2}$, justify your answer.

a) Determine the first derivative.

b) On what intervals is $f(x)$ increasing and decreasing?

c) Find and classify all relative extrema.

d) Determine the second derivative.

e) On what intervals is the function concave up/down?

f) Determine the location of any inflection points.

Solution:

a) $f'(x) = \frac{(2x-2)(x-2)-(x^2-2x+4)(1)}{(x-2)^2}$

$$f'(x) = \frac{2x^2-6x+4-x^2+2x-4}{(x-2)^2}$$

$$f'(x) = \frac{x^2-4x}{(x-2)^2}$$

b) The derivative will be equal to 0 when its numerator is equal to 0.

$$x^2-4x=0$$

$$x(x-4)=0$$

$$x=0,\ 4$$

The derivative will be undefined when its denominator is equal to 0.

$$(x-2)^2=0$$

$$x=2$$

With critical numbers $x=0,\ 2,\ 4$, use test values $x=-1,\ 1,\ 3,\ 5$ in the first derivative.

$$f'(-1) = \frac{(-1)^2-4(-1)}{(-1-2)^2} = \frac{1+4}{9} = \frac{5}{9} > 0$$

$$f'(1) = \frac{1^2-4(1)}{(1-2)^2} = \frac{1-4}{1} = -3 < 0$$

$$f'(3) = \frac{3^2 - 4(3)}{(3-2)^2} = \frac{9-12}{1} = -3 < 0$$

$$f'(5) = \frac{5^2 - 4(5)}{(5-2)^2} = \frac{25-20}{9} = \frac{5}{9} > 0$$

From the signs of these test values, we can say that

$f'(x) > 0$ on $(-\infty,0) \cup (4,\infty)$, so $f(x)$ is increasing on $(-\infty,0) \cup (4,\infty)$

$f'(x) < 0$ on $(0,4)$, so $f(x)$ is decreasing on $(0,4)$

c) Because $f(x)$ is increasing on $(-\infty,0) \cup (4,\infty)$ and deceasing on $(0,4)$, $f(x)$ has a relative maximum at $x = 0$, a relative minimum at $x = 4$.

d) $$f''(x) = \frac{(2x-4)(x-2)^2 - (x(x-4))(2(x-2))}{((x-2)^2)^2}$$

$$f''(x) = \frac{(2x-4)(x-2)^2 - 2x(x-4)(x-2)}{(x-2)^4}$$

$$f''(x) = \frac{(2x-4)(x-2) - 2x(x-4)}{(x-2)^3}$$

$$f''(x) = \frac{2(x-2)^2 - 2x(x-4)}{(x-2)^3}$$

$$f''(x) = \frac{2x^2 - 8x + 8 - 2x^2 + 8x}{(x-2)^3}$$

$$f''(x) = \frac{8}{(x-2)^3}$$

e) Set $f''(x) = 0$ to find inflection points.

$$\frac{8}{(x-2)^3} = 0$$

The second derivative is undefined at $x = 2$, so this is the only possible inflection point. Use test values of $x = 0, 3$ in the second derivative.

$$f''(0) = \frac{8}{(0-2)^3} = \frac{8}{-8} = -1 < 0$$

$$f''(3) = \frac{8}{(3-2)^3} = \frac{8}{1} = 8 > 0$$

From the signs of these test values, we can say that

$f''(x) < 0$ on $(-\infty, 2)$, so $f(x)$ is concave down on $(-\infty, 2)$

$f''(x) > 0$ on $(2, \infty)$, so $f(x)$ is concave up on $(2, \infty)$

f) While the function $f(x)$ changes concavity at $x = 2$, from concave down on the left to concave up on the right, $f(x)$ is undefined at $x = 2$, so it can't have an inflection point there. There are no inflection points.

■ 12. A rectangular box is made from a sheet of tin that is 4 in. by 4 in. Congruent squares are cut from the corners and the sides are folded up to form the box. What height of the box will produce the maximum volume?

Solution:

The volume of the box will be

$$V(x) = x(4-2x)(4-2x)$$

$$V(x) = x(16-16x+4x^2)$$

$$V(x) = 16x - 16x^2 + 4x^3$$

Find critical points.

$$V'(x) = 16 - 32x + 12x^2$$

$$12x^2 - 32x + 16 = 0$$

$$3x^2 - 8x + 4 = 0$$

$$(3x - 2)(x - 2) = 0$$

$$x = \frac{2}{3}, 2$$

The critical point $x = 2$ doesn't make sense. If we cut a 2×2 square from each corner of the 4×4 sheet, we'd cut out the entire sheet. So $x = \frac{2}{3}$ is the only critical number we need to consider. Use the second derivative test to classify $x = \frac{2}{3}$.

$$V''(x) = 24x - 32$$

$$V''\left(\frac{2}{3}\right) = 24\left(\frac{2}{3}\right) - 32$$

$$V''\left(\frac{2}{3}\right) = 16 - 32$$

$$V''\left(\frac{2}{3}\right) = -16 < 0$$

So the critical point $x = \frac{2}{3}$ is associated with a local maximum, which means that the volume of the box is maximized when $x = \frac{2}{3}$. The box's height is given by x, so the height that maximizes the box is $\frac{2}{3}$ inches.