5.1 USING THE MEAN VALUE THEOREM

1. Say whether the Mean Value Theorem applies for each function over the associated interval. If it doesn't apply, explain why not.

Function	Interval	Does MVT apply?
$f(x) = 5x^2 - 3x + 1$	$[1,3]$	The MVT applies
$g(x) = \dfrac{1}{(x-1)^2}$	$[0,2]$	The MVT doesn't apply because $g(x)$ is not continuous at $x = 1$
$h(x) = \dfrac{x+3}{x-2}$	$[5,10]$	The MVT applies
$m(x) = x^{\frac{2}{3}}$	$[-8,8]$	The MVT doesn't apply because $m(x)$ is not differentiable at $x = 0$
$p(x) = x^{\frac{1}{2}}$	$[0,8]$	The MVT applies

2. Find the value $x = c$ over the interval $[1,4]$ where average rate of change on the interval is equal to the instantaneous rate of change of $f(x) = x + \dfrac{4}{x}$.

Find $f'(x)$, then evaluate it at $x = c$.

$$f(x) = x + 4x^{-1}$$

$$f'(x) = 1 - 4x^{-2}$$

$$f'(c) = 1 - 4c^{-2}$$

Then the Mean Value Theorem says

$$f'(c) = \frac{f(x) - f(a)}{x - a} \quad \to \quad 1 - 4c^{-2} = \frac{f(4) - f(1)}{4 - 1}$$

$$1 - \frac{4}{c^2} = \frac{4 + \frac{4}{4} - \left(1 + \frac{4}{1}\right)}{4 - 1} \quad \to \quad 1 - \frac{4}{c^2} = \frac{4 + 1 - (1 + 4)}{3}$$

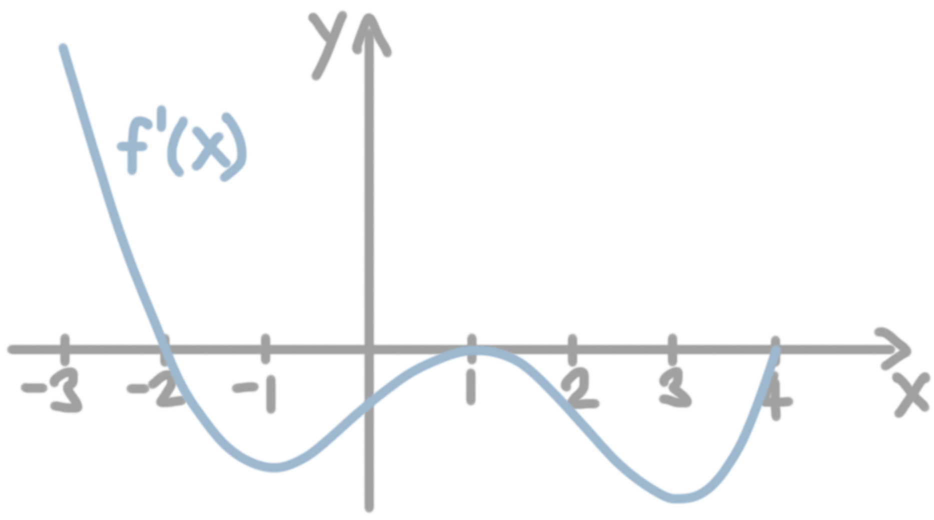
$$1 - \frac{4}{c^2} = \frac{0}{3} \quad \to \quad \frac{4}{c^2} = 1 \quad \to \quad 4 = c^2 \quad \to \quad c = \pm 2$$

Only $c = 2$ lies in the interval $[1,4]$, so $c = x = 2$ is the point in the interval at which the instantaneous rate of change of the function is equal to the average rate of change over the interval.

3. The Mean Value Theorem applies when a function which is continuous on an interval $[a, b]$ and differentiable on (a, b).

5.2 EXTREME VALUE THEOREM, GLOBAL VERSUS LOCAL EXTREMA, AND CRITICAL POINTS

1. The graph below shows the derivative f' of a function f that's differentiable on the interval $[-3,4]$.



a. Is f guaranteed to have an absolute maximum and an absolute minimum over the closed interval? Explain.

Yes, because f is differentiable, it must also be continuous. And because we're looking at closed interval, the Extreme Value Theorem applies.

b. List all critical points of f in the interval $[-3,4]$.

$x = -2,\ 1,\ 4$

2. For $f(x) = ax^3 + bx^2 + cx + d$, find the values of a, b, c, and d such that $f(x)$ has critical values at $x = 4$ and $x = -2$, and that $f(0) = 1$ and $f'(0) = 8$.

Apply the initial condition $f(0) = 1$ to the original function.

$$1 = a(0)^3 + b(0)^2 + c(0) + d$$

$$1 = d$$

Find the derivative, $f'(x)$, then apply the initial condition $f'(0) = 8$.

$$f'(x) = 3ax^2 + 2bx + c$$

$$8 = 3a(0)^2 + 2b(0) + c$$

$$8 = c$$

Then the original function and the derivative become

$$f(x) = ax^3 + bx^2 + 8x + 1$$

$$f'(x) = 3ax^2 + 2bx + 8$$

The function has critical values at $x = 4$ and $x = -2$, so

$$3a(4)^2 + 2b(4) + 8 = 0 \quad \to \quad 48a + 8b + 8 = 0 \quad \to \quad 6a + b = -1$$

$$3a(-2)^2 + 2b(-2) + 8 = 0 \quad \to \quad 12a - 4b + 8 = 0 \quad \to \quad 3a - b = -2$$

Add these equations to cancel b.

$$6a + b + (3a - b) = -1 + (-2) \quad \to \quad 6a + b + 3a - b = -1 - 2$$

$$9a = -3 \quad \to \quad a = -\frac{1}{3}$$

Then

$$6\left(-\frac{1}{3}\right) + b = -1 \quad \to \quad -2 + b = -1 \quad \to \quad b = 1$$

So $a = -\frac{1}{3}$, $b = 1$, $c = 8$, and $d = 1$.

3. Say whether each of the following statements is true or false.

a. The slope of a function is 0 at a global extrema.

False. The slope may be 0, but it doesn't have to be.

b. The slope of a function is 0 at a local extrema.

False. The slope may be 0, but it could also be undefined.

c. A local extrema can be a global extrema.

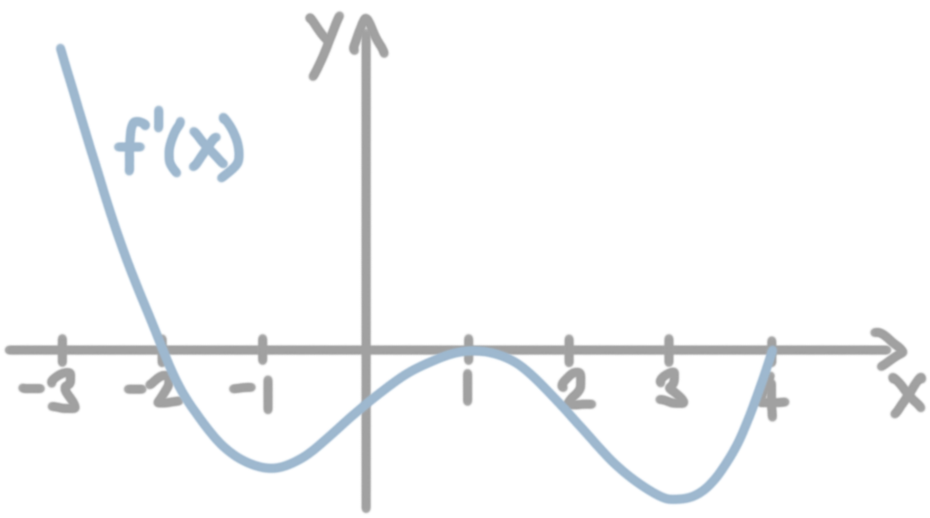
True. Quadratic functions are an example, in which the single local extrema is also the global extrema.

d. There can be more than one global maximum.

False. A curve can reach it's absolute maximum in multiple locations (think about the graph of $y = \sin x$), but the absolute maximum is still a single value, given by the maximum height that the function attains in its domain.

5.3 DETERMINING INTERVALS ON WHICH A FUNCTION IS INCREASING OR DECREASING

1. The graph below shows the graph of the derivative of a differentiable function f on the interval $[-3,4]$. Using the graph, fill in the table about $f(x)$.



Interval	f' positive or negative?	f increasing or decreasing?
(-3,-2)	positive	increasing
(-2,1)	negative	decreasing
(1,4)	negative	decreasing

2. For $f(x) = x^2 - 4x + 3$, say where $f(x)$ is increasing and decreasing, and explain how you know.

Find the derivative, $f'(x)$, then solve for critical points.

$$f'(x) = 2x - 4$$

$$2x - 4 = 0 \quad \to \quad 2x = 4 \quad \to \quad x = 2$$

Test values $x = 0,\ 3$ in the first derivative.

$$f'(0) = 2(0) - 4 = -4 < 0$$

$$f'(3) = 2(3) - 4 = 2 > 0$$

Because $f'(x) < 0$ to the left of $x = 2$, $f(x)$ is decreasing on $(-\infty,2)$. And Because $f'(x) > 0$ to the right of $x = 2$, $f(x)$ is increasing on $(2,\infty)$.

5.4 USING THE FIRST DERIVATIVE TEST TO DETERMINE RELATIVE (LOCAL) EXTREMA

1. Match each statement with the appropriate justification.

a. f has a relative maximum at $x = a$.

$f'(a) = 0$ and $f'(x)$ is changing from positive to negative at $x = a$.

b. f has a relative minimum at $x = a$.

$f'(a) = 0$ and $f'(x)$ is changing from negative to positive at $x = a$.

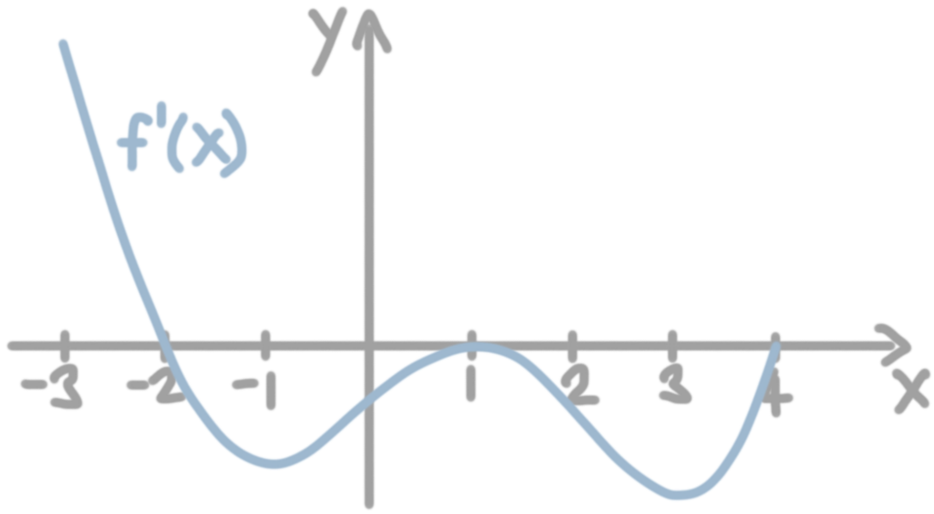
c. f is increasing in the interval (a, b).

$f'(x) > 0$ on the interval (a, b).

d. f is decreasing along the interval (a, b).

$f'(x) < 0$ for (a, b).

2. The graph below shows the derivative of a differentiable function f on the interval $[-3,4]$. Say which statement(s) below are true.



I. $f(x)$ has a relative maximum at $x = 1$ because the graph of $f'(x)$ changes from increasing to decreasing at $x = 1$.

False. Because $f'(x)$ intersects the x-axis at $x = 1$, $f(x)$ has a possible critical point at $x = 1$. But because the graph of $f'(x)$ is negative (below the x-axis) to the left and right of $x = 1$, $f(x)$ doesn't change direction, and therefore has no relative extrema at $x = 1$.

II. $f(x)$ has a relative minimum at $x = -2$ because the graph of $f'(x)$ changes from positive to negative at $x = -2$.

False. A relative minimum occurs in $f(x)$ when the graph of $f'(x)$ changes from negative to positive, but at $x = -2$, the graph of $f'(x)$ changes from positive to negative, indicating a relative maximum for $f(x)$ at that point.

III. $f(x)$ has a relative maximum at $x = -2$ because the graph of $f'(x)$ changes from positive to negative at $x = -2$.

True. When the graph of $f'(x)$ changes from positive to negative, it indicates a relative maximum for $f(x)$ at the same point.

3. The function $f(x) = x^4 - 4x^3$ has two critical values, $x = 0$ and $x = 3$. Fill in each cell of the the table below with "positive" or "negative," then use the table to classify $x = 0$ and $x = 3$ as either a relative maximum, relative minimum, or neither. Justify your answer.

	x<0	0<x<3	x>3
f'(x)	Negative	Negative	Positive

The critical number $x = 0$ doesn't represent a relative maximum or a relative minimum, because the sign of $f'(x)$ doesn't change at that point.

The critical number $x = 3$ represents a relative minimum because $f'(3) = 0$ and $f'(x)$ changes sign from negative to positive at that point.

4. The following table gives information about a differentiable function $g(x)$. Say whether each statement below is true or false.

	-2	0	2	5
g'(x)	3	0	-3	4

a. $g(x)$ has a relative maximum at $x = 0$.

False. We can't tell from the table that $g'(x) > 0$ for the entire interval $x = [-2,0]$, nor can we tell that $g'(x) < 0$ for the entire interval $x = [0,2]$, so we can't say with certainty that $g(x)$ has a relative maximum when $x = 0$.

b. $g(x)$ has at least two relative extrema.

True. It's possible that there are more than two relative extrema, but we can see two sign changes for $g'(x)$ from the table, which means there are at least two relative extrema.

c. $x = 0$ is a critical point of $g(x)$.

True. Because $g'(0)=0$, $x=0$ is a critical point of $g(x)$.

d. $x=5$ is a relative maximum of $g(x)$.

False. $g'(5)$ is not a critical value, so $x=5$ cannot be the location of a relative maximum.

5. Fill in the blanks with one of the following terms: positive, negative, increasing, or decreasing.

a. At a relative minimum on $f(x)$, $f(x)$ changes from ___decreasing___ to ___increasing___. At a relative minimum on $f(x)$, $f'(x)$ changes from ___negative___ to ___positive___.

b. At a relative maximum on $f(x)$, $f(x)$ changes from ___increasing___ to ___decreasing___. At a relative maximum on $f(x)$, $f'(x)$ changes from ___positive___ to ___negative___.

5.5 USING THE CANDIDATES TEST TO DETERMINE ABSOLUTE (GLOBAL) EXTREMA

1. For $f(x)=x^3-3x^2$ on the interval $[-1,5]$, fill in the table below with all four of the candidates for absolute extrema, and then identify the absolute maximum and absolute minimum values.

Find the derivative, then use it to find critical points.

$$f'(x)=3x^2-6x$$

$$3x^2-6x=0 \quad \to \quad x^2-2x=0 \quad \to \quad x(x-2)=0 \quad \to \quad x=0,\ 2$$

Then the candidates (including the critical points and the endpoints of the interval), along with their corresponding values,

$$f(-1)=(-1)^3-3(-1)^2=-1-3=-4$$

$$f(0)=0^3-3(0)^2=0-0=0$$

$$f(2)=2^3-3(2)^2=8-12=-4$$

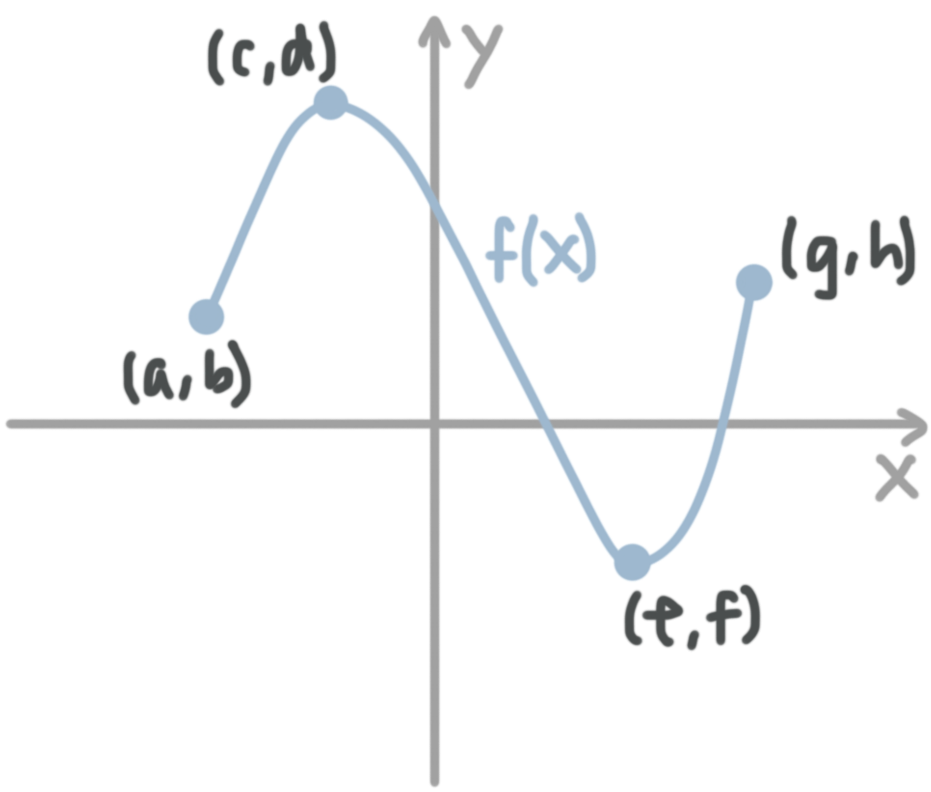
$$f(5)=5^3-3(5)^2=125-75=50$$

are given in the table.

x	-1	0	2	5
f(x)	-4	0	-4	50

From the values associated with each candidate, we can say that, in the interval $[-1,5]$, the function has an absolute maximum at $(5,50)$ and an absolute minimum at of -4, which occurs at $(-1,-4)$ and $(2,4)$.

2. The graph of $f(x)$ is shown below on the interval $[-3,4]$. Fill in the blanks.



On the given interval, the function has candidates for extrema at:

- $x = a$
- $x = c$
- $x = e$
- $x = g$

In the interval, $f(x)$ has its absolute maximum at $x = c$ and its absolute minimum at $x = e$.

5.6 DETERMINING CONCAVITY OF FUNCTIONS OVER THEIR DOMAINS

1. For $g''(x) = x^2(x-3)(x+4)$, fill in each cell of the table with "positive" or "negative," or "0." Then use the table to answer each question.

Evaluate $g''(x)$ at a test value in each interval.

$$g''(-5) = (-5)^2(-5-3)(-5+4) = 25(-8)(-1) = 200 > 0$$

$$g''(-1) = (-1)^2(-1-3)(-1+4) = (-4)(3) = -12 < 0$$

$$g''(1) = 1^2(1-3)(1+4) = (-2)(5) = -10 < 0$$

$$g''(4) = 4^2(4-3)(4+4) = 16(1)(8) = 128 > 0$$

Therefore, the completed table is

	x<-4	-4<x<0	0<x<3	x>3
g''(x)	Positive	Negative	Negative	Positive

Then we can say

$g(x)$ is concave down where $g''(x) < 0$, or at $(-4,3)$.

$g(x)$ is concave up where $g''(x) > 0$, or at $(-\infty, -4) \cup (3,\infty)$

$g(x)$ has inflection points at $x = -4$ and $x = 3$, since these are the points where the function changes concavity. $x = 0$ is not an inflection point because $g(x)$ is concave down to the immediate left and right of $x = 0$, and therefore the function has no change in concavity at that point.

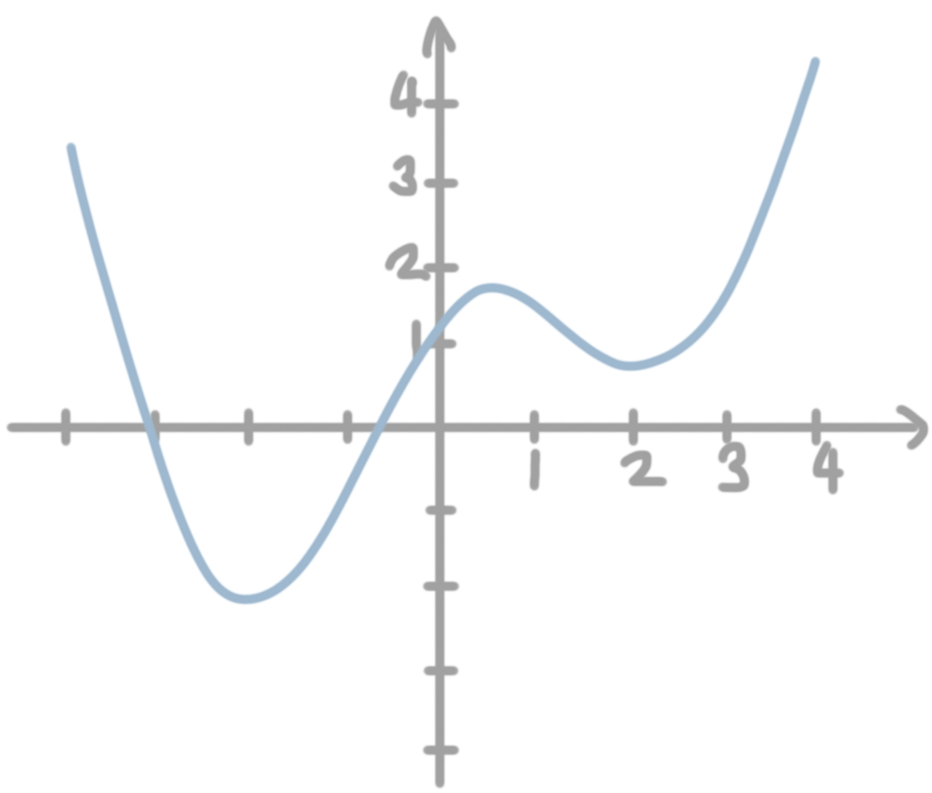
2. For $f'(x) = x^4 - 4x^2$, identify the following:

Intervals where $f(x)$ is concave up: $(-\sqrt{2},0) \cup (\sqrt{2}, \infty)$

Intervals where $f(x)$ is concave down: $(-\infty, -\sqrt{2}) \cup (0,\sqrt{2})$

Number of inflection points: 3 inflection points

3. Use the graph of $f(x)$ below to fill in each cell of the table with "concave up" or "concave down."



x	-2	-1	1	2
f(x)	concave up	concave up	concave down	concave up

5.7 USING THE SECOND DERIVATIVE TEST TO DETERMINE EXTREMA

1. Use the information given in the table below to mark each of the following statements as true or false.

x	-3	0	3
f'(x)	0	-2	0
f''(x)	-1	0	5

a. $f(x)$ is concave down at $x = -3$. True, $f''(-3) < 0$.

b. $f(x)$ is increasing at $x = 0$. False, $f'(0) < 0$.

c. $f(x)$ has a relative maximum at $x = 0$. False, $f'(0) = -2$, so it's not a critical point, and there's no relative extrema at $x = 0$.

d. $f(x)$ has a relative maximum at $x = 3$. False, while $f'(3) = 0$, $f''(3) > 5$, meaning $f(x)$ is concave up at $x = 3$, and therefore that $x = 3$ is the location of a relative minimum.

e. $f(x)$ has a relative maximum at $x = -3$. True. $f''(x) > 0$ at the critical point $x = -3$, so $x = -3$ is the location of a relative maximum.

2. For $f(x) = x^4 - 4x^3$, there are two critical values, $x = 0$ and $x = 3$. Complete the following table, then fill in the blanks in the statement below.

x	0	3
f'(0)	0	0
f''(x)	0	36

From the table, we know $x = 0$ is a inconclusive ("relative maximum," "relative minimum," or "inconclusive") because $f'(0)$ is 0 and $f''(0)$ = (<, =, or >) 0. $x = 3$ is a relative minimum ("relative maximum," "relative minimum," or" inconclusive") because $f'(0)$ is 0 and $f''(0)$ $>$ (<, =, or >) 0.

3. Put the following steps in order so that they describe step-by-step how to conduct the second derivative test.

1. Take the first derivative of $f(x)$

3. Take the second derivative of $f(x)$

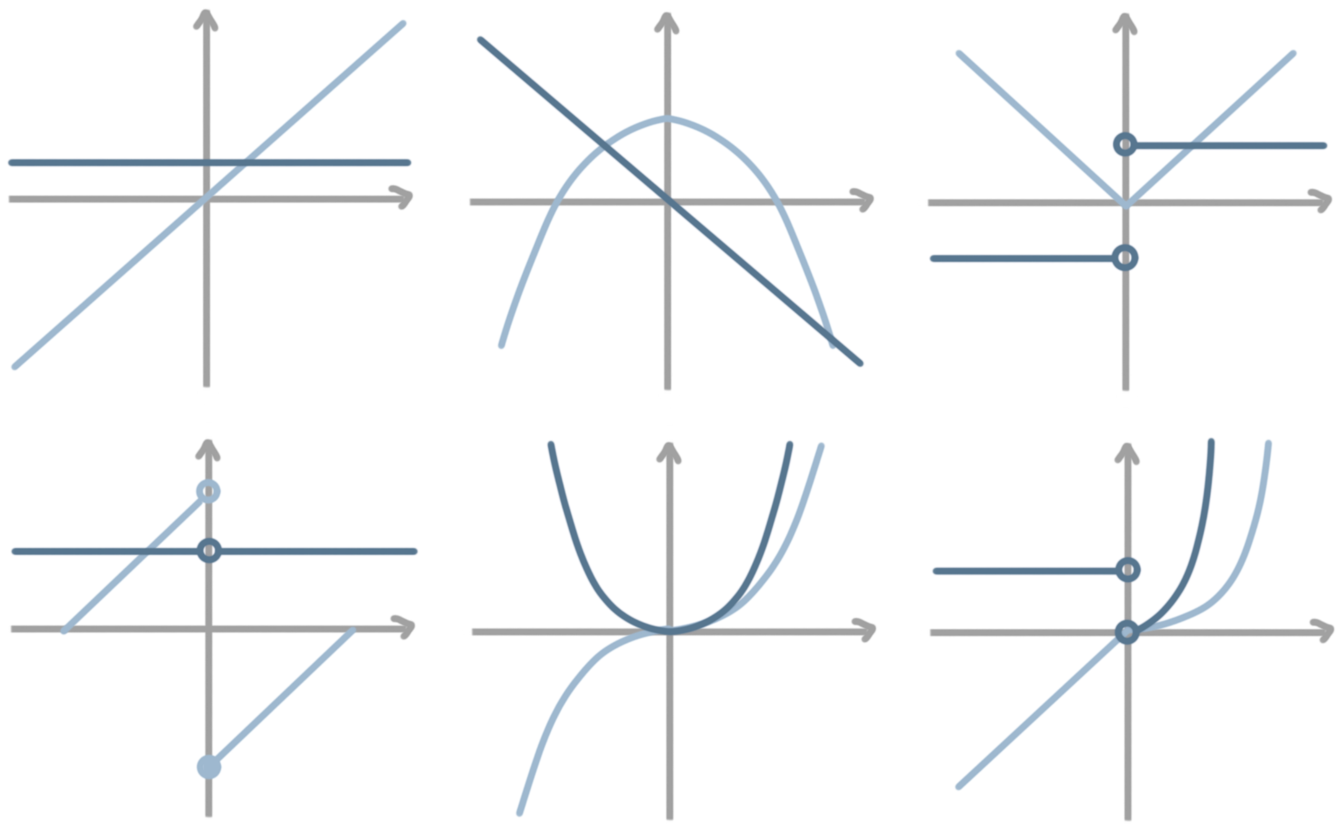
4. Plug the critical values of into $f''(x)$

5. Analyze the sign of the results

2. Find critical values at which the first derivative is 0 or undefined

5.8 SKETCHING GRAPHS OF FUNCTIONS AND THEIR DERIVATIVES

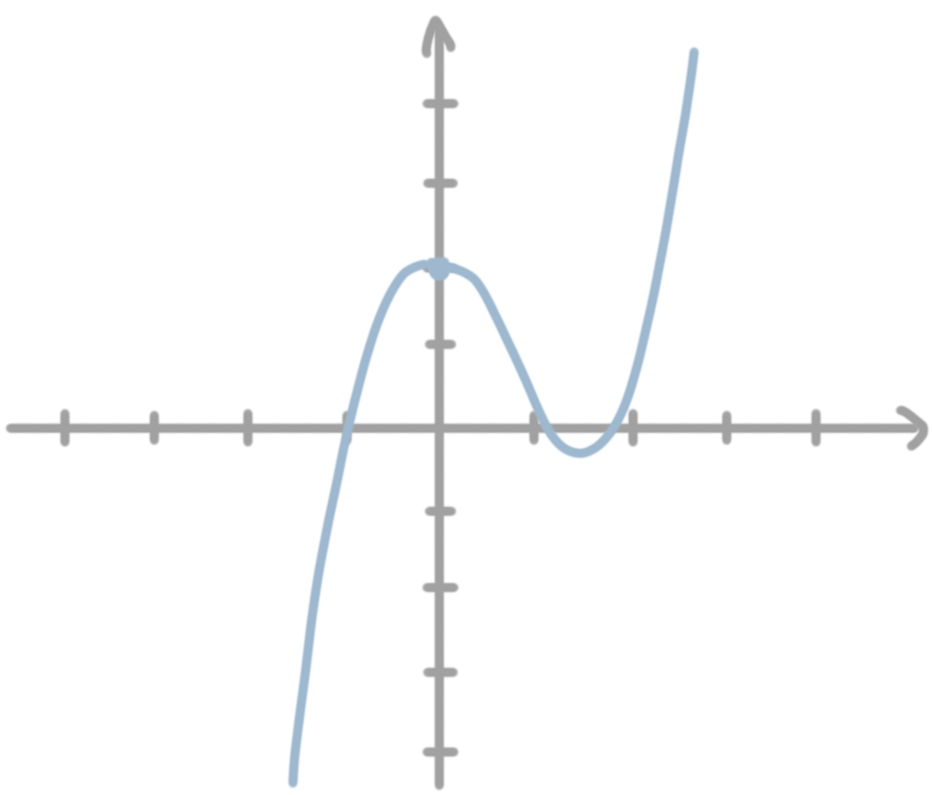
1. For each graph below, sketch a possible graph of its derivative.



2. Sketch a graph of the continuous function $f(x)$ that meets the criteria below over the interval $[-3,3]$.

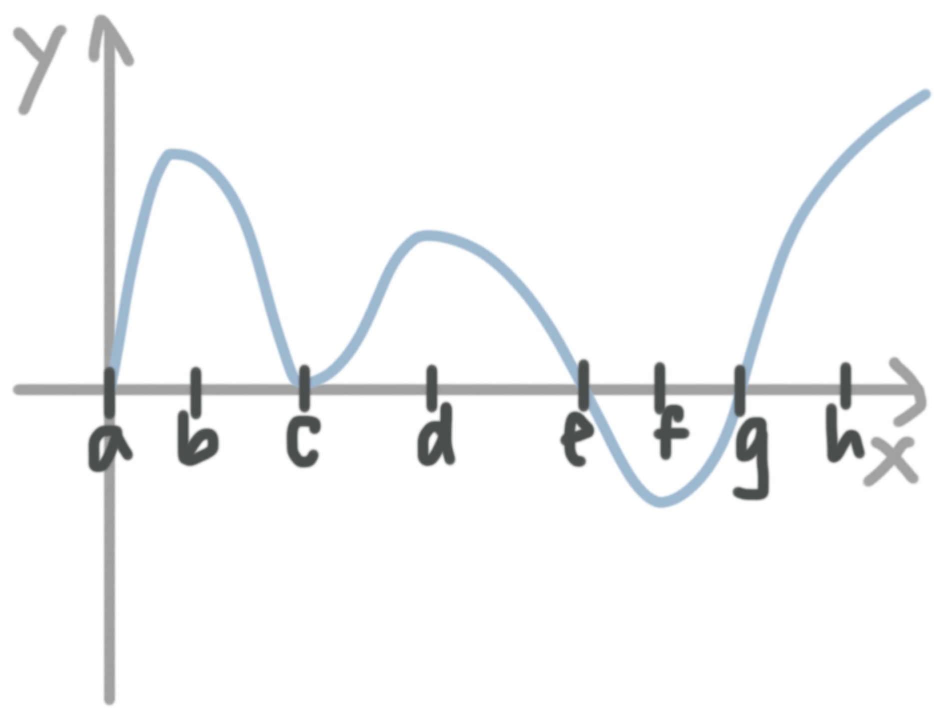
$f(0) = 2$ $\qquad f''(0) < 0$ $\qquad f'(2) > 0$ $\qquad f'(-1) > 0$ $\qquad f'(0) = 0$

One option could be:



5.9 CONNECTING A FUNCTION, ITS FIRST DERIVATIVE, AND ITS SECOND DERIVATIVE

1. Below is the graph of $g'(x)$ over the closed interval $[a, h]$. Answer the following questions about the graph of $g(x)$, and justify your answers.



a. List each value of x at which $g(x)$ has a relative maximum.

$x = e$, because $g'(e) = 0$ and $g'(x)$ changes from positive to negative at $x = e$.

b. List each value of x at which $g(x)$ has an inflection point.

$x = b,\ c,\ d,\ f$, because $g'(x)$ changes direction at these points, either from increasing to decreasing or decreasing to increasing

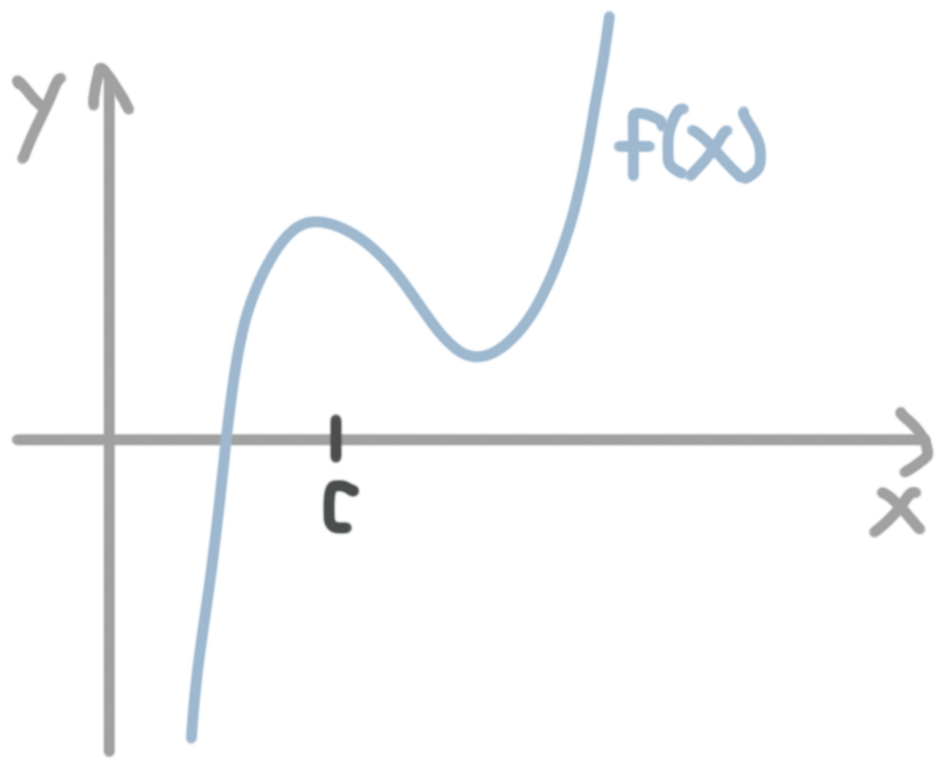
c. Over which interval(s) is $g(x)$ concave up?

$(a, b) \cup (c, d) \cup (f, h)$, because $g'(x)$ is increasing over these intervals

d. Over which interval(s) is $g(x)$ decreasing?

(e, g), because $g'(x)$ is negative over this interval

2. The graph of $f(x)$ below is a differentiable function with a horizontal tangent line at $x = c$. Which of the statements below about the relationship between $f(c)$, $f'(c)$, and $f''(c)$ is true?



A $\quad f(c) < f'(c) < f''(c)$

B $\quad f(c) < f''(c) < f'(c)$

C $\quad f'(c) < f''(c) < f(c)$

D $\quad f''(c) < f'(c) < f(c)$

Answer choice D is true. The function $f(x)$ is above the x-axis at $x = c$, so $f(c) > 0$. Because there's a horizontal tangent line at $x = c$, we know $f'(c) = 0$. And because the curve is concave down at $x = c$, we know $f''(c) < 0$. So negative < 0 < positive, or $f''(c) < f'(c) < f(c)$.

3. Answer each of the questions below.

a. Give two ways to justify the existence of a relative maximum.

We can either 1) use the first derivative test to show a change in $f'(x)$ from positive to negative, or we can 2) use the second derivative test to show that $f''(x)$ is negative at a critical value.

b. How would you use the graph of $f''(x)$ to show that $f(x)$ is concave up?

Wherever $f''(x)$ is positive (above the x-axis), $f(x)$ will be concave up.

c. How would you use the graph of $f'(x)$ to show that $f(x)$ is concave up?

Wherever $f'(x)$ is increasing, $f(x)$ is concave up.

d. What would you look for on the graph of $f''(x)$ in order to find an inflection point of $f(x)$?

Wherever $f''(x)$ is crossing the x-axis, in either direction, $f(x)$ will have an inflection point.

e. What would you look for on the graph of $f'(x)$ in order to find an inflection point on $f(x)$?

Wherever $f'(x)$ is changing direction, either from increasing to decreasing or from decreasing to increasing, $f(x)$ will have an inflection point.

5.10 INTRODUCTION TO OPTIMIZATION PROBLEMS

1. For what value of x does $f(x)$ have a relative minimum, given $f'(x) = (x+1)^2(x-3)$?

$x = 3$, because $f'(3) = 0$ and $f'(x)$ changes from negative to positive at $x = 3$

2. The height of a soccer ball being kicked around on a field can be tracked with by function $h(t) = -t^2 + 25t$, with $h(t)$ measured in feet and t measured in seconds. Find the maximum height of the soccer ball.

Find the derivative $h'(t) = -2t + 25$, then find critical points.

$$-2t + 25 = 0$$

$$-2t = -25$$

$$t = \frac{-25}{-2} = 12.5$$

The second derivative is $h''(t) = -2$, so the second derivative test gives $h''(12.5) = -2 < 0$, indicating that $t = 12.5$ gives a maximum height. The maximum height is

$$h(12.5) = -12.5^2 + 25(12.5)$$

$$h(12.5) = -156.25 + 312.5$$

$$h(12.5) = 156.25 \text{ feet}$$

5.11 SOLVING OPTIMIZATION PROBLEMS

1. Find two numbers with maximum product that sum to 52.

The sum is

$$x + y = 52 \quad \to \quad x = 52 - y$$

Then we'll say the product is

$$P = xy \quad \to \quad P = (52 - y)y \quad \to \quad P = 52y - y^2$$

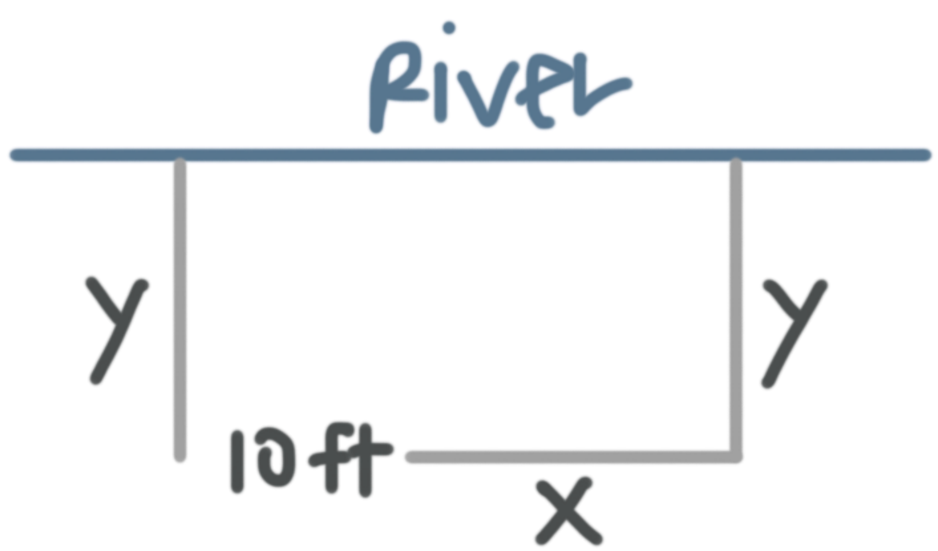
Find the derivative and solve for critical values.

$$P' = 52 - 2y$$

$$52 - 2y = 0 \quad \to \quad 52 = 2y \quad \to \quad y = 26$$

So $x = 52 - y = 52 - 26 = 26$. The two numbers are $(x, y) = (26,26)$.

2. A farmer wants to use 800 feet of fencing to enclose a rectangular cow pasture. He plans to use a straight river as one of the sides of the rectangle and leave a 10-foot wide space for a gate. Find the dimensions of the pasture that will enclose the maximum area by answering each of the questions below.



a. Write down the function the farmer needs to maximize.

$$A = lw$$

$A = (x+10)y$

b. Write a constraint equation.

The farmer only has 800 feet of fence, so the perimeter is constrained. From the diagram, the perimeter is given by $800 = x + 2y$, or $x = 800 - 2y$.

c. Use the constraint equation to write the optimization equation in terms of one variable.

$A = (800 - 2y + 10)y$

$A = (810 - 2y)y$

$A = 810y - 2y^2$

d. Find critical points.

$A'(y) = 810 - 4y$

$0 = 810 - 4y$

$y = 202.5$

e. What are the values of x and y that maximize the area of the pasture?

If $y = 202.5$, then $x = 800 - 2(202.5) = 395$. So the area of the pasture is maximized when the dimensions are 405 ft $\times$ 202.5 ft.

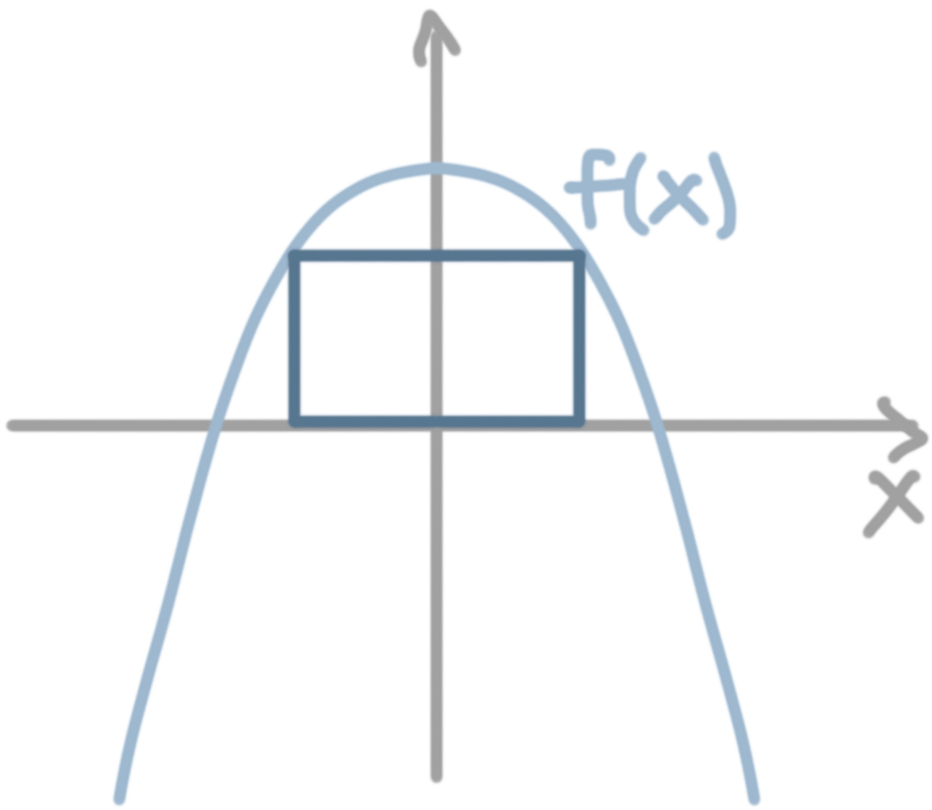
f. Find the maximum area of the pasture.

Plug the maximizing dimensions back into the area function.

$A = (395 + 10)(202.5)$

$$A = 82{,}012.5 \text{ ft}^2$$

2. The graph shows $f(x) = 9 - x^2$ around an inscribed rectangle. The base of the rectangle sits on the x-axis, with its upper vertices on $f(x)$. Find the maximum area of the rectangle?



Let the base of the rectangle be given by $2x$ and the height of the rectangle by $f(x) = 9 - x^2$. Then the area of the rectangle is

$$A = 2x(9 - x^2) \quad \to \quad A = 18x - 2x^3$$

Find critical points.

$$A' = 18 - 6x^2$$

$$18 - 6x^2 = 0 \quad \to \quad 18 = 6x^2 \quad \to \quad 3 = x^2 \quad \to \quad x = \pm\sqrt{3}$$

Then maximum area is given by

$$A = 18\sqrt{3} - 2(\sqrt{3})^3 \quad \to \quad A = 18\sqrt{3} - 2(3\sqrt{3}) \quad \to \quad A = 18\sqrt{3} - 6\sqrt{3}$$

$A = 12\sqrt{3}$

5.12 EXPLORING BEHAVIORS OF IMPLICIT RELATIONS

1. The table below relates to some function $y = f(x)$. Fill in the blanks in the statement below with "positive," "negative," "zero," "increasing," "decreasing," or "constant."

(x,y)	(-2,3)	(0,2)	(6,3)
dy/dx	-1/6	0	1/6

At $(-2,3)$, y is decreasing, since dy/dx is negative. At $(0,2)$, y is constant, since dy/dx is zero. At $(6,3)$, y is increasing, since dy/dx is positive.

2. For some $y = f(x)$, read the table below and then say whether each statement below is true or false.

(x,y)	(-1,3)	(0,4)	(3,5)
dy/dx	Undefined	0	2
d^2y/dx^2	-3	4	0

I. At $(-1,3)$, $f(x)$ has a relative maximum.

True, by the second derivative test.

II. At $(3,5)$, $f(x)$ is concave up.

False, because the second derivative is 0, which is not a positive value.

III. Between $(-1,3)$ and $(0,4)$, $f(x)$ has an inflection point.

True, because the second derivative is changing sign.

IV. At $(0,4)$, $f(x)$ has a vertical tangent line.

False, because a horizontal tangent line exists only when $dy/dx = 0$.