■ 1. The height of a tree over time can be modeled by the function $h(x) = 0.5t + \frac{2}{4t+1}$, where h is measured in feet and t in years. What is the average height of the tree over the first 10 years?

[A]	2.686 ft	B	6.214 ft
C	12.427 ft	D	4.598 ft

Solution: A

Use the average value formula.

$$\frac{1}{b-a}\int_a^b f(x)\ dx$$

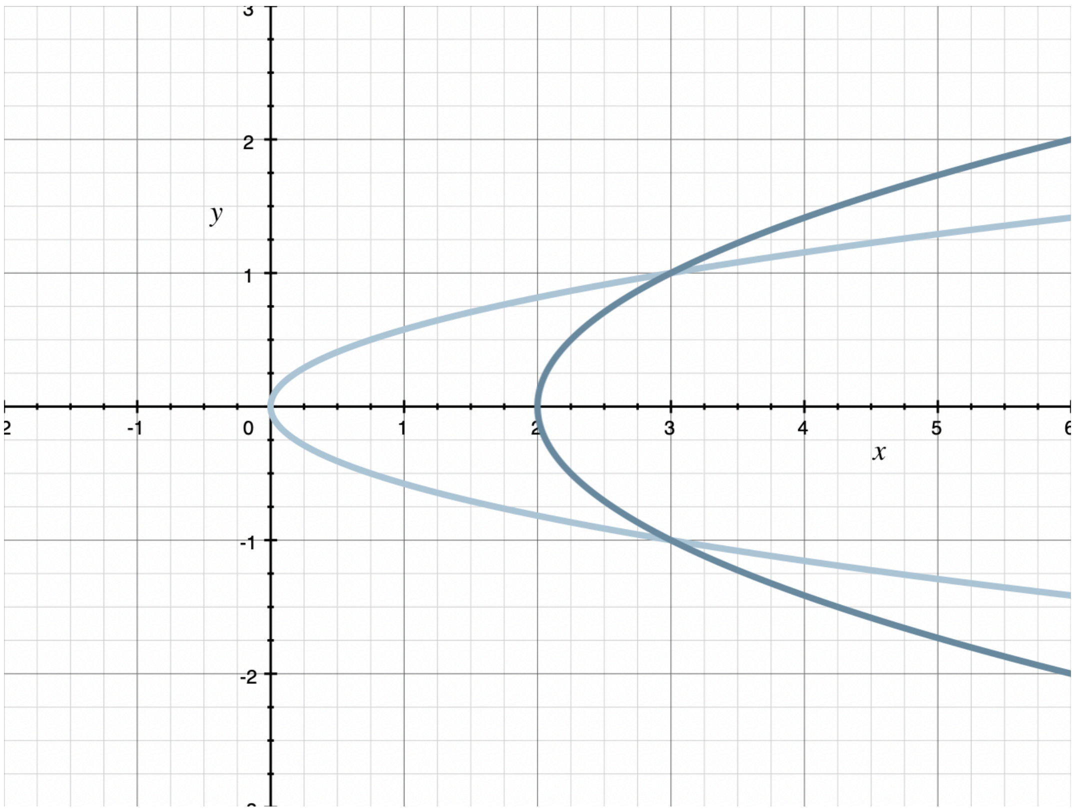
$$\frac{1}{10-0}\int_0^{10} 0.5t + \frac{2}{4t+1}\ dt$$

$$\frac{1}{10}\left(0.25t^2 + 0.5\ln|4t+1|\right)\Big|_0^{10}$$

$$\frac{1}{10}\left(0.25(10)^2 + 0.5\ln|4(10)+1|\right) - \frac{1}{10}\left(0.25(0)^2 + 0.5\ln|4(0)+1|\right)$$

$$2.5 + 0.05\ln 41 \approx 2.686 \text{ ft}$$

■ 2. Find the area between the curves $x = 3y^2$ and $x = y^2 + 2$, shown in the graph below.



A $\frac{8}{3}$

B $\frac{4}{3}$

C $-\frac{4}{3}$

D $-\frac{8}{3}$

Solution: A

To find points of intersection, set the curves equal to one another, then solve for y.

$$3y^2 = y^2 + 2 \quad \to \quad 2y^2 = 2 \quad \to \quad y^2 = 1 \quad \to \quad y = \pm 1$$

These two points of intersection define the endpoints of the interval in terms of y, which means our next step is to determine which curve has a larger x-value on the y-interval $[-1,1]$.

We can do this by picking a y-value within the interval and plugging it into both functions. Whichever curve returns a larger value we'll call $f(y)$, and whichever curve returns a lower value we'll call $g(y)$.

Plugging $y=0$ into both functions, we get

$$x=3y^2=3(0)^2=0$$

$$x=y^2+2=(0)^2+2=2$$

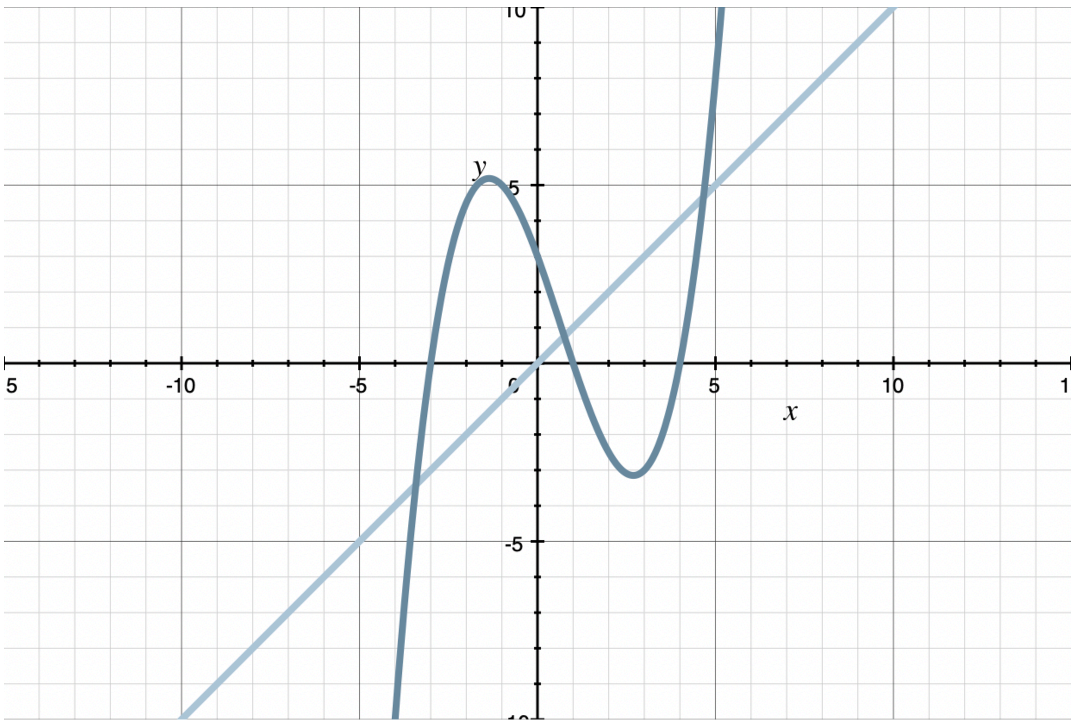
Since $x=y^2+2$ gives a larger value, we'll say $g(y)=3y^2$ and $f(y)=y^2+2$. Now we can plug these functions and the interval we found earlier into the formula for area between left and right curves.

$$\int_a^b f(y)-g(y)\ dy \quad \to \quad \int_{-1}^1 y^2+2-3y^2\ dy \quad \to \quad \int_{-1}^1 -2y^2+2\ dy$$

$$\frac{-2y^3}{3}+2y\Big|_{-1}^{1}=\frac{-2(1)^3}{3}+2(1)-\left(\frac{-2(-1)^3}{3}+2(-1)\right)$$

$$\frac{-2}{3}+2-\frac{2}{3}+2 \quad \to \quad -\frac{4}{3}+\frac{12}{3} \quad \to \quad \frac{8}{3}$$

■ 3. Given the graph of $f(x)=x$ and $g(x)=\frac{1}{4}x^3-\frac{1}{2}x^2-\frac{11}{4}x+3$ below, which intersect at $x=-3.417$, 0.752 and 4.664, which of the following integral expressions could be used to find the total area between these curves?



A $\int_{-3.417}^{4.664} f(x) - g(x)\ dx$

B $\int_{-3.417}^{0.752} g(x) - f(x)\ dx + \int_{0.752}^{4.664} f(x) - g(x)\ dx$

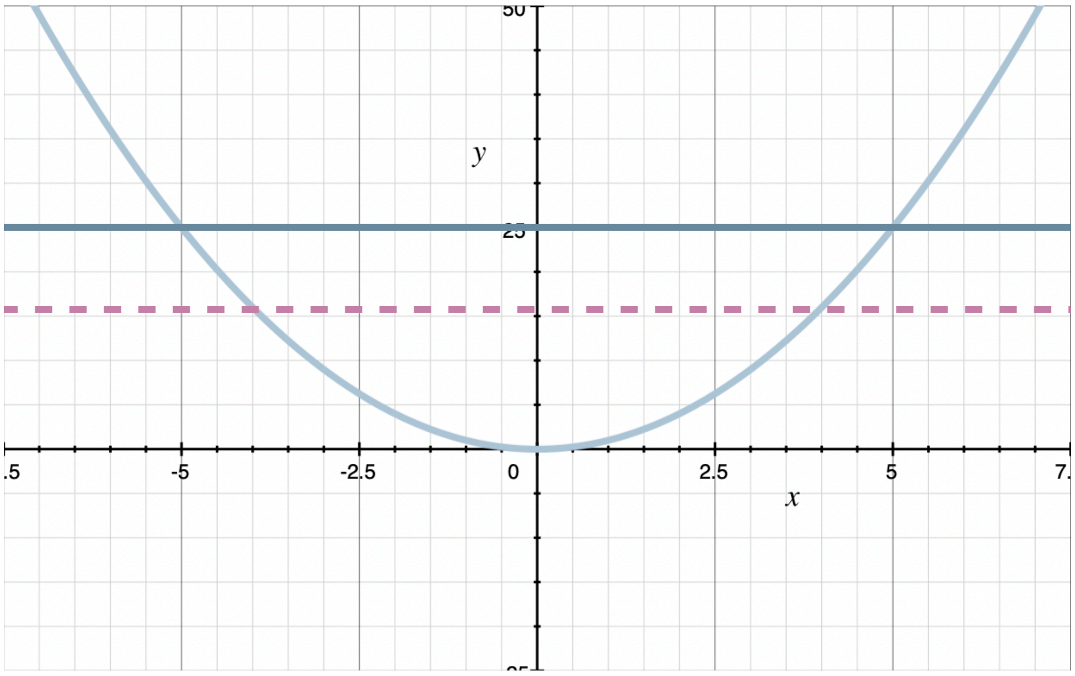
C $\int_{-3.417}^{4.664} |f(x) + g(x)|\ dx$

D $2\int_{0.752}^{4.664} g(x) - f(x)\ dx$

Solution: B

The curves enclose two regions. In the first, $g(x) > f(x)$, and in the second, $f(x) > g(x)$. The integral that reflects this is answer choice B.

■ 4. The horizontal line $y = k$ divides the area bounded by the curves $f(x) = x^2$ and $g(x) = 25$ into two equal parts. Find k.



A $k = \sqrt{\frac{25\sqrt[3]{2}}{2}}$

B $k = -\frac{25\sqrt[3]{2}}{2}$

C $k = -\sqrt{\frac{25\sqrt[3]{2}}{2}}$

D $k = \frac{25\sqrt[3]{2}}{2}$

Solution: D

The enclosed region spans $[-5,5]$, so the area of the entire region is given by

$$\int_{-5}^{5} 25 - x^2\ dx \quad \to \quad 25x - \frac{1}{3}x^3\Big|_{-5}^{5}$$

$$25(5) - \frac{1}{3}(5)^3 - \left(25(-5) - \frac{1}{3}(-5)^3\right) \quad \to \quad 125 - \frac{125}{3} - \left(-125 + \frac{125}{3}\right)$$

$$125-\frac{125}{3}+125-\frac{125}{3} \quad \to \quad \frac{500}{3}$$

Half the area of this region is $\frac{250}{3}$. Notice from the graph that since the functions are $y=x^2$ and $y=k$, the points of intersection are $(-\sqrt{k},k)$ and $(\sqrt{k},k)$. Therefore, the bounds of integration will be $[-\sqrt{k},\sqrt{k}]$.

$$A=\int_{-\sqrt{k}}^{\sqrt{k}} k-x^2\ dx \quad \to \quad A=kx-\frac{1}{3}x^3\Big|_{-\sqrt{k}}^{\sqrt{k}}$$

$$A=k\sqrt{k}-\frac{1}{3}(\sqrt{k})^3-\left(k(-\sqrt{k})-\frac{1}{3}(-\sqrt{k})^3\right)$$

$$A=k^{\frac{3}{2}}-\frac{1}{3}k^{\frac{3}{2}}-\left(-k^{\frac{3}{2}}+\frac{1}{3}k^{\frac{3}{2}}\right) \quad \to \quad A=k^{\frac{3}{2}}-\frac{1}{3}k^{\frac{3}{2}}+k^{\frac{3}{2}}-\frac{1}{3}k^{\frac{3}{2}}$$

$$A=\frac{4}{3}k^{\frac{3}{2}}$$

Then, because this area is equal to $\frac{250}{3}$, we can find k.

$$\frac{4}{3}k^{\frac{3}{2}}=\frac{250}{3} \quad \to \quad 4k^{\frac{3}{2}}=250 \quad \to \quad 16k^3=250^2$$

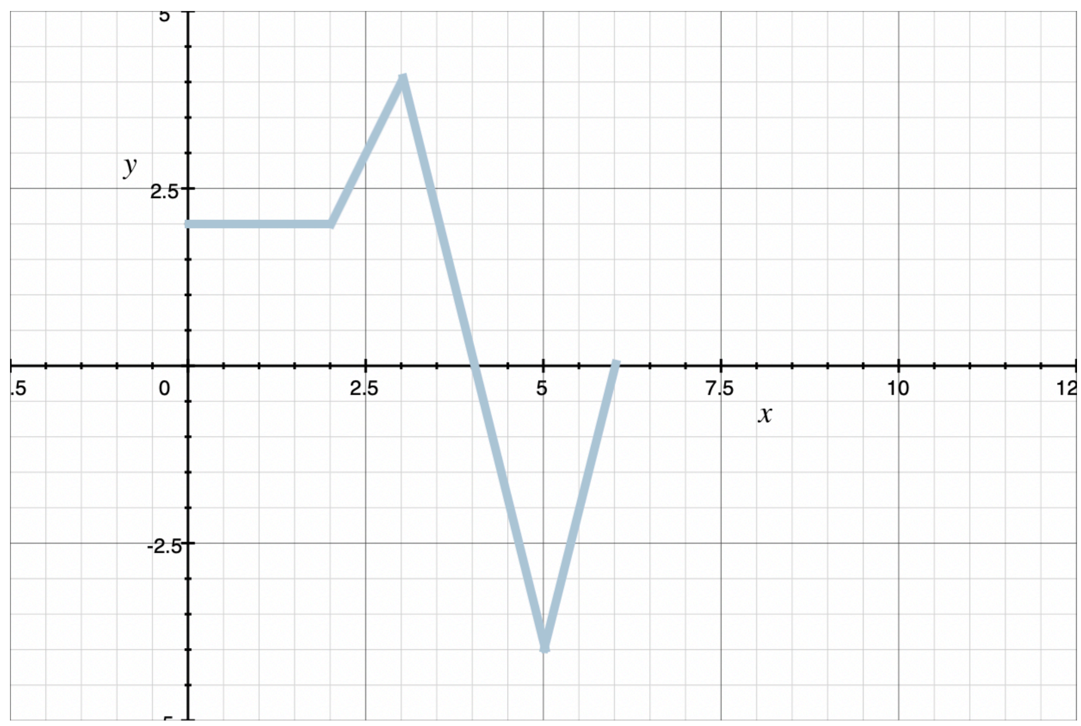
$$k^3=\frac{250^2}{16} \quad \to \quad k=\sqrt[3]{\frac{250^2}{16}}$$

$$k=\sqrt[3]{\frac{25\cdot 25\cdot 10\cdot 10}{16}} \quad \to \quad k=\sqrt[3]{\frac{25\cdot 25\cdot 5\cdot 5}{4}} \quad \to \quad k=\frac{25}{\sqrt[3]{4}}$$

Rationalize the denominator.

$$k = \frac{25\sqrt[3]{16}}{4} \quad \to \quad k = \frac{25\sqrt[3]{8}\sqrt[3]{2}}{4} \quad \to \quad k = \frac{25\sqrt[3]{2}}{2}$$

■ 5. Given that the graph of $f(x)$ below consists of line segments, what is the average value of $f(x)$ on the interval $[0,6]$?



A $\frac{5}{2}$

B $\frac{13}{2}$

[C] $\frac{5}{6}$

D $\frac{13}{6}$

Solution: C

First, we'll find the area of each region.

Area on $[0,2] = 4$

Area on $[2,3]=2+0.5(1)(2)=3$

Area on $[3,4]=0.5(1)(4)=2$

Area on $[4,6]=0.5(2)(-4)=-4$

Then the average value is

$$\frac{1}{b-a}\int_a^b f(x)\ dx \quad \to \quad \frac{1}{6-0}\int_0^6 f(x)\ dx$$

$$\frac{1}{6}(4+3+2-4) \quad \to \quad \frac{5}{6}$$

■ 6. Given that the velocity of a particle moving along the y-axis is $v(t)=0.3t-1+2\cos t$, where $v(t)$ is measured in ft/sec, use a calculator to find the total distance travelled by the particle during its first 10 seconds of motion.

A 3.912 ft

B 0.322 ft

C 24.899 ft

D 12.746 ft

Solution: D

Total distance travelled over an interval is the integral of the absolute value of the velocity.

$$\text{distance}=\int_0^{10}|0.3t-1+2\cos t|\ dt\approx 12.746$$

■ 7. The rate of change of the depth of a river, where depth D is measured in feet and time t is measured in days, can be modeled by $D'(t)=(6t-t^2)\sin\left(\frac{t}{12}\right)$. Use a calculator to determine how much the depth of the water changes from day 0 to day 8.

A Decreases by 9.894 ft

B Decreases by -1.237 ft

C Increases by 0.411 ft

D Increases by 17.290 ft

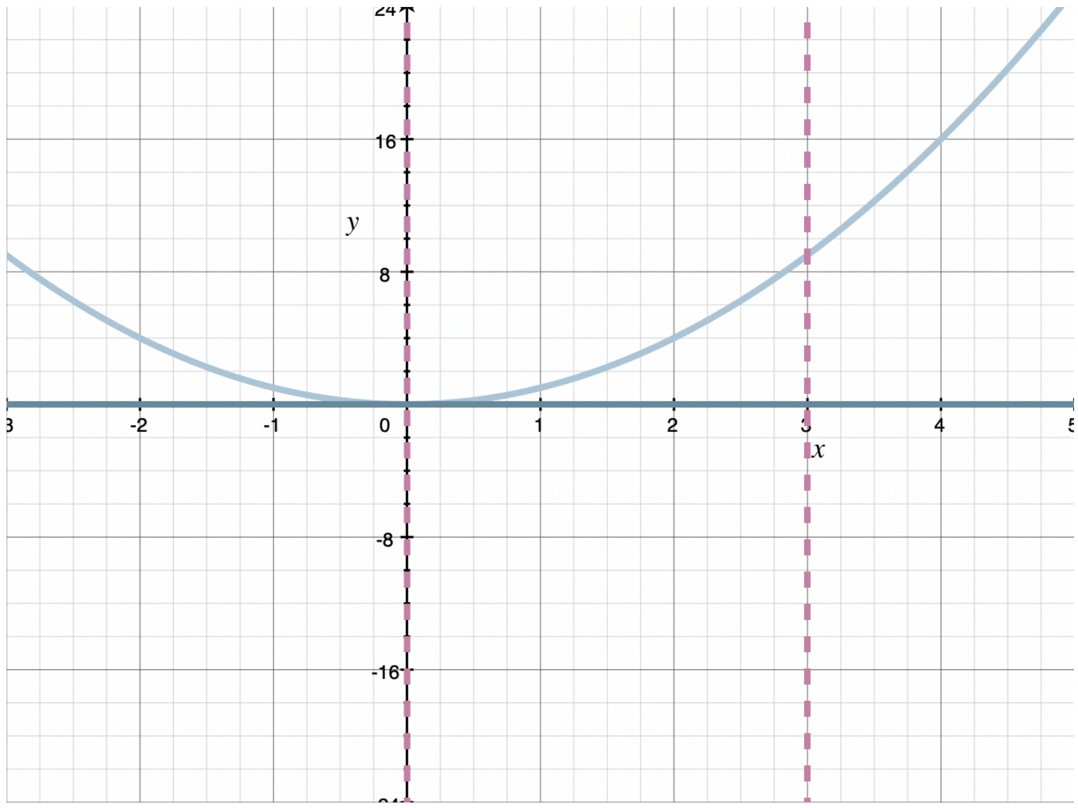
Solution: C

The integral gives

$$\int_0^8 (6t-t^2)\sin\left(\frac{t}{12}\right)\approx 0.411$$

Because the result is positive, the depth of the water increases over that 8-day period of time.

■ 8. Use disks to find the volume of the solid formed by rotating the region enclosed by the curves $y=x^2$, $y=0$, $x=0$, and $x=3$ about the x-axis.



A $V=\frac{243}{5}\pi$ cubic units

B $V=\frac{243}{5}$ cubic units

C $V=243\pi$ cubic units

D $V=81\pi$ cubic units

Solution: A

Because we're rotating about the x-axis, and because our slices of volume must always be perpendicular to the axis of rotation, that means we'll be taking vertical slices of volume.

Which means that the width of each infinitely thin slice of volume can be given by dx, which means we'll be integrating with respect to x. Therefore, the limits of integration will be given by $x=[0,3]$. The outer radius will be defined by $y=x^2$.

$$V = \int_a^b \pi \left[f(x) \right]^2 \ dx \quad \to \quad V = \int_0^3 \pi (x^2)^2 \ dx \quad \to \quad V = \int_0^3 \pi x^4 \ dx$$

Integrate, then evaluate over the interval.

$$V = \frac{1}{5} \pi x^5 \Big|_0^3 \quad \to \quad V = \frac{1}{5} \pi (3)^5 - \left(\frac{1}{5} \pi (0)^5 \right) \quad \to \quad V = \frac{243\pi}{5}$$

■ 9. The region bounded by the graphs of $y = x^2 + 2$, $y = 4$, and the y-axis is revolved about the line $x = 3$. Use a calculator to ind the volume of the solid of revolution.

A 9.314

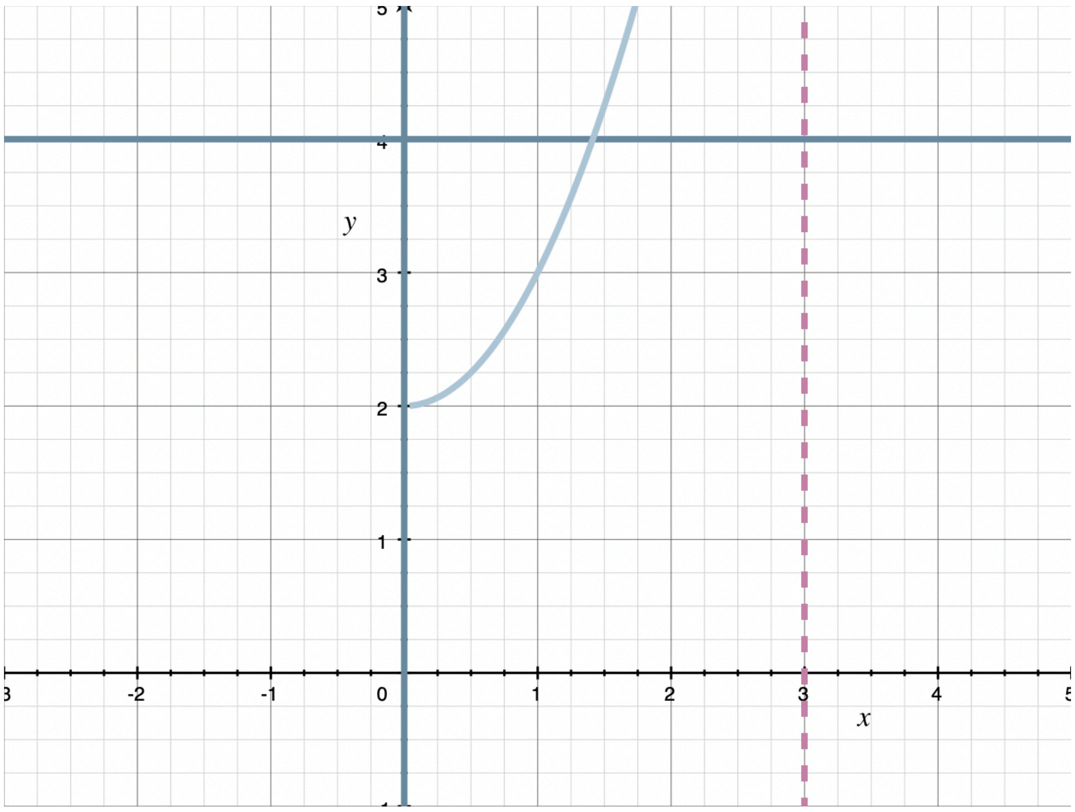
B 29.260

C 6.283

D 83.837

Solution: B

A sketch of the region is



We'll use horizontal slices of volume. The outer radius is the distance from the axis of rotation to the far edge of one of these rectangles, $k-g(y)=3-0=3$.

The inner radius is the distance from the axis to the near edge of the rectangle. We convert the equation $y=x^2+2$ to $x=\sqrt{y-2}$, and get the inner radius as $k-f(y)=3-\sqrt{y-2}$. Then the volume of revolution is

$$V=\int_c^d \pi\left[k-g(y)\right]^2-\pi\left[k-f(y)\right]^2\ dy$$

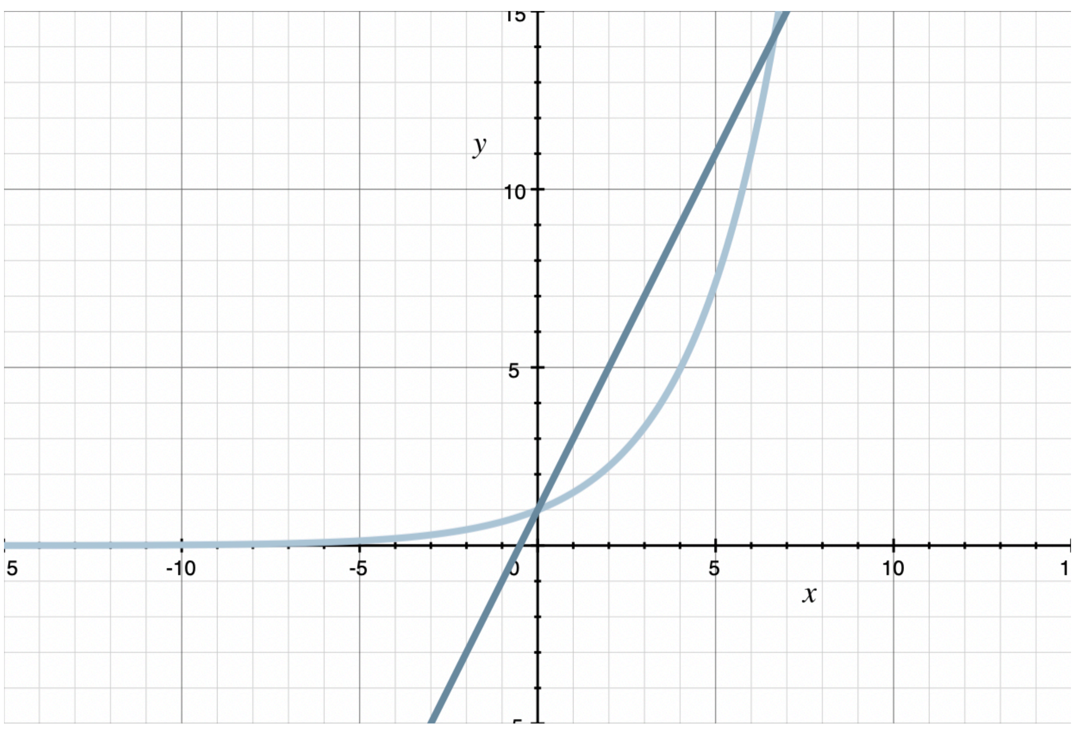
$$V=\int_2^4 3^2\pi-(3-\sqrt{y-2})^2\pi\ dy$$

$$V=\int_2^4 9\pi-(9-6\sqrt{y-2}+(y-2))\pi\ dy$$

$$V=\int_2^4 9\pi-9\pi+6\pi\sqrt{y-2}-(y-2)\pi\ dy$$

$$V=\int_2^4 6\pi\sqrt{y-2}-\pi y+2\pi\ dy\approx 29.260$$

■ 10. Consider the functions $f(x)=e^{0.4x}$ and $g(x)=2x+1$. If a solid is constructed with the area between $f(x)$ and $g(x)$ as its base, and if cross sections of the solid perpendicular to the x-axis are semicircles, then use a calculator to find the volume of the solid.



A 22.247

B 44.496

C 91.488

D 177.982

Solution: A

The base of one cross section is a vertical rectangle perpendicular to the x-axis. The height of the rectangle is given by $h=(2x+1)-e^{0.4x}$.

The area of one cross section, which is a semicircle, will be $A = \frac{1}{2}\pi r^2$. The radius of the semicircle is half the height h, giving an area of

$$A = \frac{1}{2}\pi\left(\frac{h}{2}\right)^2 = \frac{\pi}{8}h^2$$

Integrate the area formula from 0 to the intersection at 6.651.

$$V = \frac{\pi}{8}\int_0^{6.651} \left[(2x+1) - (e^{0.4x})\right]^2 \ dx \approx 22.247$$

■ 11. Consider the region bounded by the graphs of $y = x(6 - x)$ and $y = x \sin x$.

a. Use a calculator to determine the area between the curves.

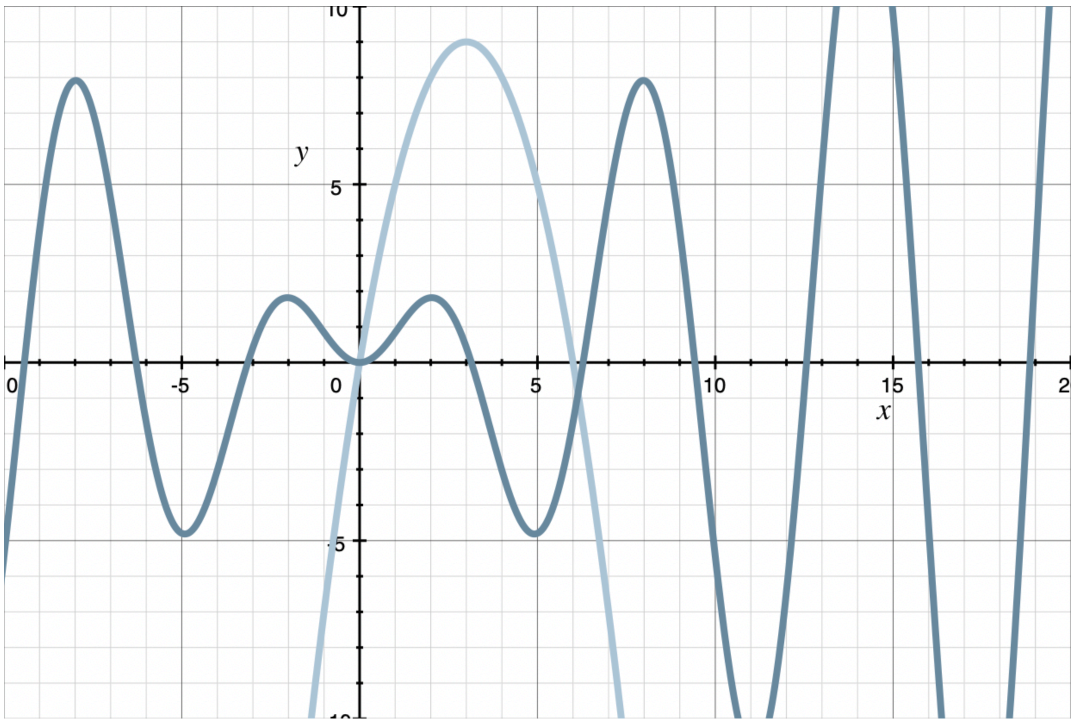
b. Find the volume of the solid generated by revolving the region around the line $y = 10$.

c. Find the volume of the solid generated by revolving the region around the line $y = -5$.

d. Find the volume of a solid with known cross sections that are rectangles perpendicular to the x-axis with a height that's three times their base.

Solution:

a. Graph the curves, consider a representative rectangle, and find the intersection points at $x = 0$ and $x = 6.141$. Then use a calculator to find area.



$$A = \int_0^{6.141} x(6-x) - x\sin x \ dx \approx 42.160$$

b. When we revolve the region around the line $y = 10$, we get $r = 10 - x\sin x$ and $r = 10 - x(6-x)$. Then the volume of revolution would be

$$V = \pi \int_0^{6.141} (10 - x\sin x)^2 - (10 - x(6-x))^2 = 1{,}959.420$$

c. When we revolve the region around the line $y = -5$, we get $R = x(6-x) + 5$ and $r = x\sin x + 5$. Then the volume of revolution would be

$$V = \pi \int_0^{6.141} (x(6-x)+5)^2 - (x\sin x + 5)^2 = 2{,}014.081$$

d. Identify the base first, $b = x(6-x) - x\sin x$. Then the area of a cross section, which is a rectangle with a base of b and a height three times the base, is $A = b \cdot 3b = 3b^2$. Integrate the area to get

$$V = 3\int_0^{6.141} (x(6-x) - x\sin x)^2 \ dx = 1{,}056.462$$

■ 12. The velocity of a particle is given by $v(t)=e^t\sin t$, where $v(t)$ is measured in feet per second.

a. Write, but do not evaluate, an integral expression that is the average velocity over the interval $[0,\pi]$.

b. What is the average acceleration of the particle over the interval $\left[\frac{\pi}{2},\pi\right]$?

c. What would the integral expression $\int_0^{2\pi} v(t)\ dt$ represent in the context of the problem?

d. What would the integral expression $\int_0^{2\pi} |v(t)|\ dt$ represent in the context of the problem?

Solution:

a. Using the average value formula, we'll calculate average velocity.

$$v_{avg}=\frac{1}{\pi}\int_0^{\pi} v(t)\ dt=\frac{1}{\pi}\int_0^{\pi} e^t\sin t\ dt$$

b. Average acceleration is change in velocity over change in time, so first evaluate the velocity function and the endpoints of the interval.

$$v\left(\frac{\pi}{2}\right)=e^{\frac{\pi}{2}}\sin\left(\frac{\pi}{2}\right)=e^{\frac{\pi}{2}}(1)=e^{\frac{\pi}{2}}$$

$$v(\pi)=e^{\pi}\sin\pi=e^{\pi}(0)=0$$

So average acceleration is

$$a_{avg} = \frac{v(\pi) - v\left(\frac{\pi}{2}\right)}{\pi - \frac{\pi}{2}} = \frac{0 - e^{\frac{\pi}{2}}}{\frac{\pi}{2}} = -\frac{2e^{\frac{\pi}{2}}}{\pi}$$

c. The expression $\int_0^{2\pi} v(t)\ dt$ represents the change in position of the particle, in feet, between $t = 0$ and $t = 2\pi$ seconds.

d. The expression $\int_0^{2\pi} |v(t)|\ dt$ represents the total distance the particle traveled, in feet, between $t = 0$ and $t = 2\pi$ seconds.