8.1 FINDING THE AVERAGE VALUE OF A FUNCTION ON AN INTERVAL

1. Explain what $\frac{1}{b-a}\int_a^b f(x)\ dx$ calculates for the function f over the interval $[a,b]$. [The expression calculates the average value (height) of the function f over the interval a to b.]

2. Find the average value of $g(x)=x^2+4x-1$ over the interval $[-1,2]$.

$$\frac{1}{2-(-1)}\int_{-1}^{2} x^2+4x-1\ dx \quad \to \quad \frac{1}{2+1}\left(\frac{1}{3}x^3+2x^2-x\right)\Bigg|_{-1}^{2}$$

$$\frac{1}{3}\left(\frac{1}{3}(2)^3+2(2)^2-2\right)-\frac{1}{3}\left(\frac{1}{3}(-1)^3+2(-1)^2-(-1)\right)$$

$$\frac{8}{9}+\frac{8}{3}-\frac{2}{3}+\frac{1}{9}-\frac{2}{3}-\frac{1}{3} \quad \to \quad \frac{9}{9}+\frac{3}{3} \quad \to \quad 2$$

3. The temperature (in degrees Fahrenheit) is modeled by $F(t)=45+14\sin\left(\frac{\pi t}{12}\right)$ where t is the number of hours after $8:00$ a.m. Circle the integral below that would calculate the average temperature from $10:00$ a.m. to $2:00$ p.m.

$\frac{1}{10-2}\int_2^{10} 45+14\sin\left(\frac{\pi t}{12}\right)\ dt$ $\qquad$ $\frac{1}{2-10}\int_{10}^{2} 45+14\sin\left(\frac{\pi t}{12}\right)\ dt$

$\frac{1}{6-2}\int_6^{2} 45+14\sin\left(\frac{\pi t}{12}\right)\ dt$ $\qquad$ [✓ $\frac{1}{6-2}\int_2^{6} 45+14\sin\left(\frac{\pi t}{12}\right)\ dt$]

4. If the average value of the function f over the interval $[a,b]$ is 3, find the value of $\int_a^b f(x)\ dx$.

The value of the integral is $3b - 3a$.

$$3 = \frac{1}{b-a}\int_a^b f(x)\ dx \quad \to \quad 3(b-a) = \int_a^b f(x)\ dx \quad \to \quad 3b - 3a = \int_a^b f(x)\ dx$$

8.2 CONNECTING POSITION, VELOCITY, AND ACCELERATION OF FUNCTIONS USING INTEGRALS

1. Let a position function be represented by $s(t)$, the velocity by $v(t)$, and the acceleration by $a(t)$ over an interval $[a, b]$. Match one integral to each of the following expressions.

a. $s(b) - s(a) = \int_a^b v(t)\ dt$

b. $\frac{s(b) - s(a)}{b - a} = \frac{1}{b-a}\int_a^b v(t)\ dt$

c. $v(b) - v(a) = \int_a^b a(t)\ dt$

d. $\frac{v(b) - v(a)}{b - a} = \frac{\int_a^b a(t)\ dt}{b - a}$

e. $s(b) = s(a) + \int_a^b v(t)\ dt$

2. Circle the correct position function, $x(t)$, for a particle with velocity $v(t) = \sin t + 2$, if the position of the particle at $t = 0$ is 1.

a. $x(t) = \cos t + 2t + 2$

b. $x(t) = -\cos t + 2t$

[✓ c. $x(t) = -\cos t + 2t + 2$]

$$x(t) = \int \sin t + 2\ dt \quad \to \quad x(t) = -\cos t + 2t + C$$

$$1 = -\cos(0) + 2(0) + C \quad \to \quad 1 = -1 + C \quad \to \quad C = 2$$

$$x(t) = -\cos t + 2t + 2$$

d. $x(t) = -\cos t + 2$

8.3 USING ACCUMULATION FUNCTIONS AND DEFINITE INTEGRALS IN APPLIED CONTEXTS

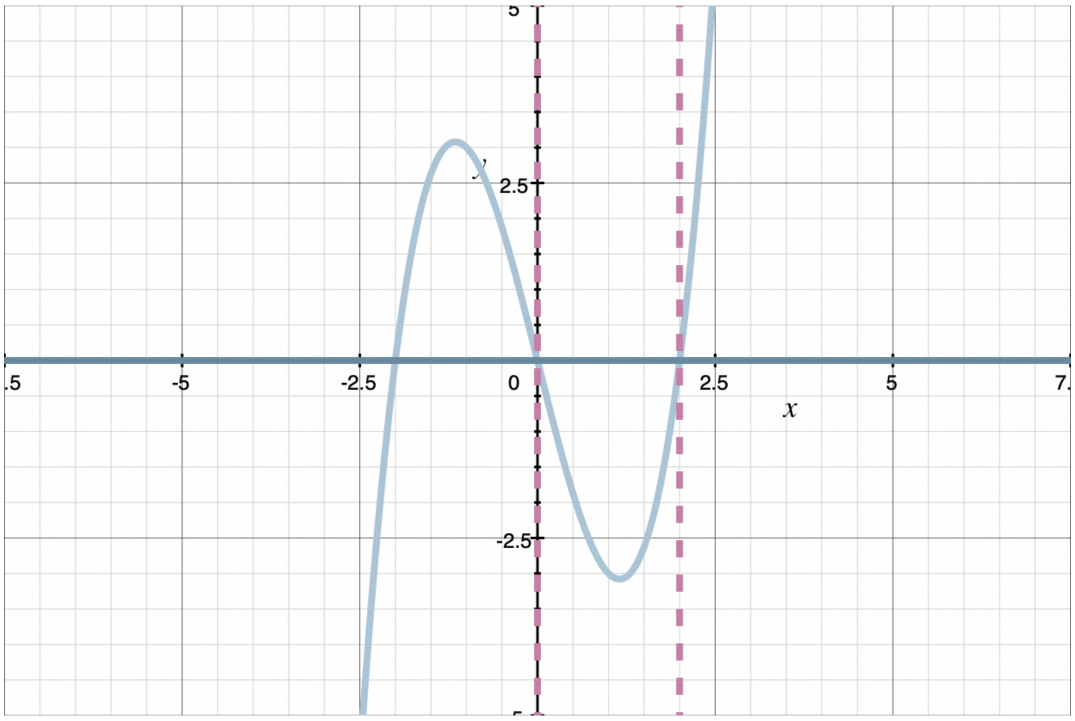
1. Karen is hiking, at a rate given by the differentiable function $m(t)$, measured in feet per minute. Match each of the following integrals with an expression below.

$\frac{1}{30}\int_0^{30} m(t)\ dt$	Karen's average rate of change over the first 30 minutes.
$\int_0^{30} m'(t)\ dt$	The change in Karen's pace over the first 30 minutes.
$\int_0^{30} m(t)\ dt$	Karen's displacement over the first 30 minutes.
$\frac{1}{30}\int_0^{30} m'(t)\ dt$	The average change in Karen's pace over the first 30 minutes.

$\int_0^{30} |m(t)|\ dt$ The total distance Karen hikes over the first 30 minutes.

8.4 FINDING THE AREA BETWEEN CURVES EXPRESSED AS FUNCTIONS OF X

1. The graph below shows the region R bounded by $y = x^3 - 4x$, $x = 0$, $y = 0$, and $x = 2$. Set up and evaluate an integral for the area of R.



We can find the area of the region R by summing the lengths of ___vertical___ (horizontal, vertical) strips, from $x =$ __0__ to $x =$ __2__. The function that defines the top of the region is ___$y = 0$___, and the function that defines the bottom is ___$y = x^3 - 4x$___. To find the area, we need to calculate the integral

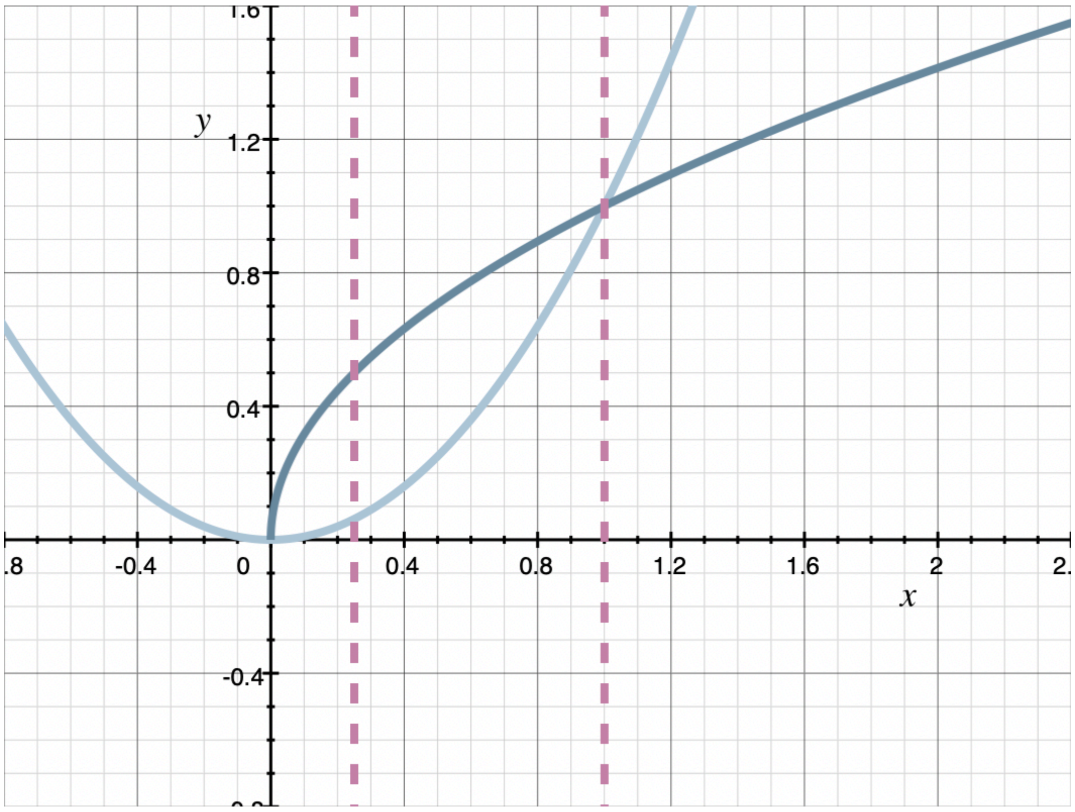
$$\int_0^2 [(\underline{\quad 0 \quad}) - (\underline{\quad x^3 - 4x \quad})]\ dx.$$

$$\int_0^2 -x^3+4x\ dx \quad \to \quad -\frac{1}{4}x^4+2x^2\Big|_0^2 \quad \to \quad -\frac{1}{4}(2)^4+2(2)^2-\left(-\frac{1}{4}(0)^4+2(0)^2\right)$$

$$-\frac{1}{4}(16)+2(4) \quad \to \quad -4+8 \quad \to \quad 4$$

2. Draw a sketch of the region R bounded by the given curves and lines below. Set up and evaluate an integral to find the area of the region R.

$$y=x^2,\ y=\sqrt{x},\ x=\frac{1}{4},\ x=1$$



We can find the area of the region by summing the lengths of ___vertical___ (horizontal, vertical) strips, from $x=$ __1/4__ to $x=$ __1__. The function that defines the top of the region is __$y=\sqrt{x}$__ and the function that defines the bottom is __$y=x^2$__. To find the area, we need to calculate the integral

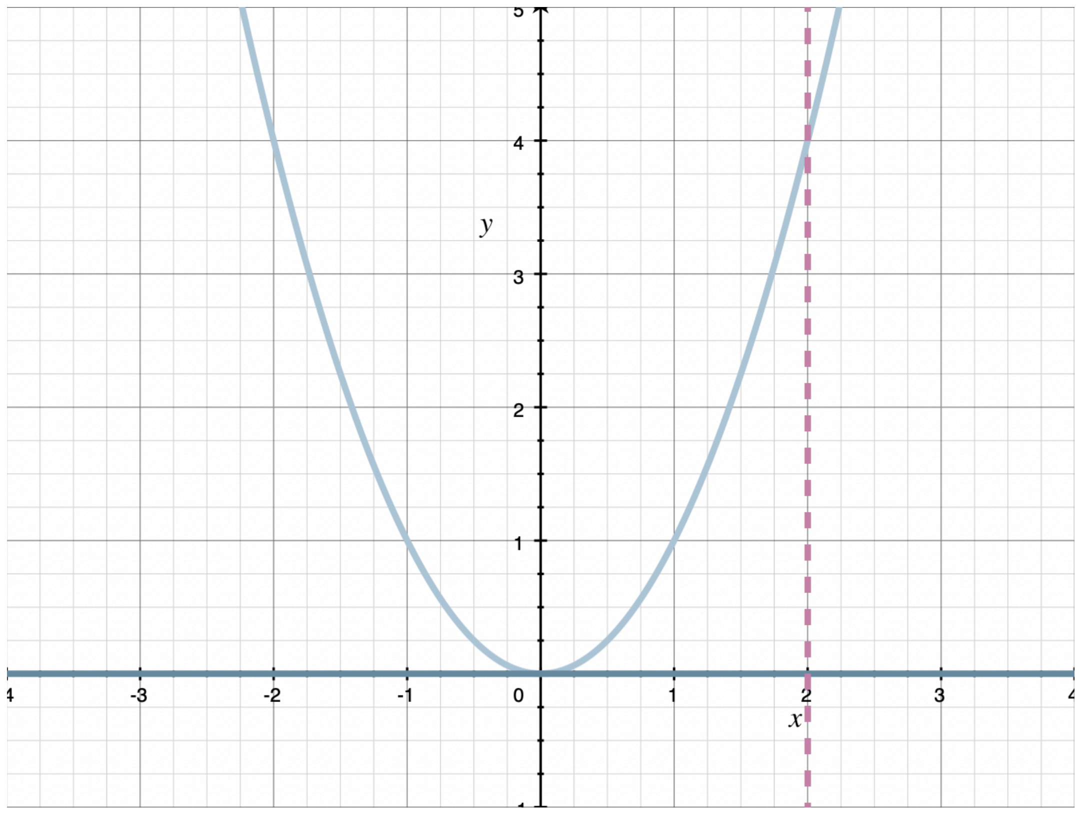
$$\int_{\frac{1}{4}}^{1}[(\underline{\quad \sqrt{x} \quad})-(\underline{\quad x^2 \quad})]\ dx.$$

$$\int_{\frac{1}{4}}^{1} \sqrt{x} - x^2\ dx \quad \to \quad \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3 \Big|_{\frac{1}{4}}^{1} \quad \to \quad \frac{2}{3}(1)^{\frac{3}{2}} - \frac{1}{3}(1)^3 - \left(\frac{2}{3}\left(\frac{1}{4}\right)^{\frac{3}{2}} - \frac{1}{3}\left(\frac{1}{4}\right)^3\right)$$

$$\frac{2}{3} - \frac{1}{3} - \frac{1}{12} + \frac{1}{192} \quad \to \quad \frac{64}{192} - \frac{16}{192} + \frac{1}{192} \quad \to \quad \frac{49}{192}$$

3. Find a vertical line $x = k$ that divides the area enclosed by $y = x^2$, $y = 0$, and $x = 2$ into two equal parts.

a. Sketch the region.



b. Set up an integral to find total area, then evaluate the integral.

$$\int_0^2 x^2\ dx = \frac{8}{3}$$

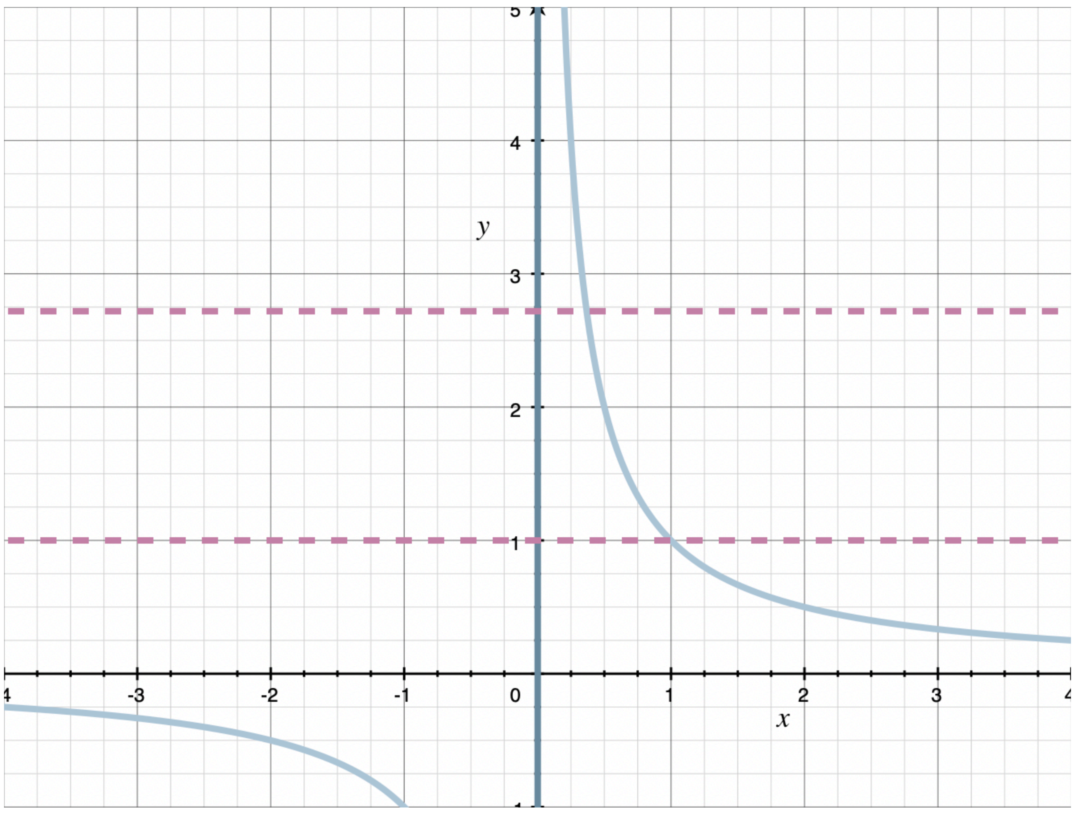
c. Find a line $y = k$ that divides the area of the region in half.

$$\int_0^k x^2\ dx = \frac{4}{3} \quad \to \quad \frac{1}{3}x^3 \Big|_0^k = \frac{4}{3} \quad \to \quad \frac{1}{3}k^3 - \frac{1}{3}(0)^3 = \frac{4}{3}$$

$$\frac{1}{3}k^3=\frac{4}{3} \quad \to \quad k^3=4 \quad \to \quad k=\sqrt[3]{4}$$

8.5 FINDING THE AREA BETWEEN CURVES EXPRESSED AS FUNCTIONS OF Y

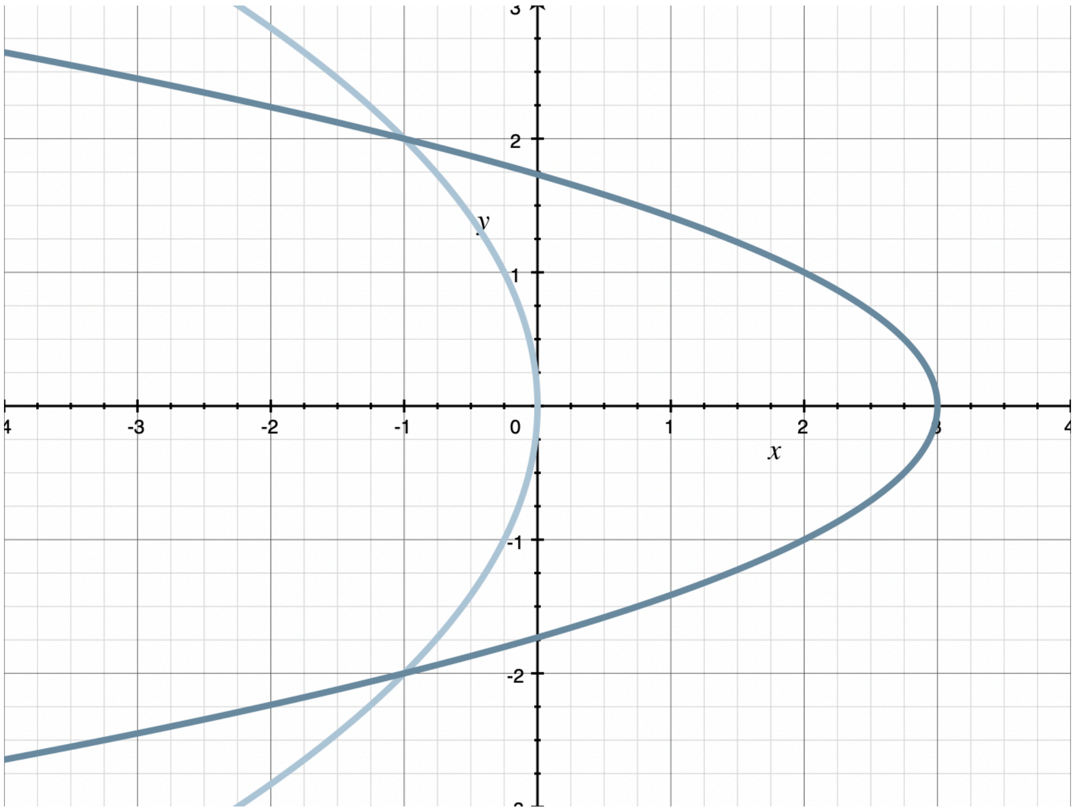
1. The graph below shows the region R bounded by $x=\frac{1}{y}$, $x=0$, $y=1$, and $y=e$. Set up and evaluate an integral that calculates the area of the region.



We can find the area of the region by summing the lengths of ___horizontal___ (horizontal, vertical) strips, from $y=1$ to $y=e$. The function that defines the right side of our region is ___$x=1/y$___, and the function that defines the left side is ___$x=0$___. To find the area, we need to calculate the integral $\int_1^e [(\underline{\quad \frac{1}{y} \quad}) - (\underline{\quad 0 \quad}$)] dy.

$$\int_1^e \frac{1}{y}\, dy \quad \to \quad \ln|y|\Big|_1^e \quad \to \quad \ln|e| - \ln|1| \quad \to \quad 1 - 0 \quad \to \quad 1$$

2. The graph below shows the region R bounded by $y^2 = -4x$ and $x = 3 - y^2$.



The points of intersection of the curves are $\underline{\quad (-1,-2) \quad}$ and $\underline{\quad (-1,2) \quad}$. We can find the area of the region by summing the lengths of $\underline{\quad \text{horizontal} \quad}$ (horizontal, vertical) strips, from $\underline{\quad y = -2 \quad}$ to $\underline{\quad y = 2 \quad}$. The function that defines the right side of the region is $\underline{\quad x = 3 - y^2 \quad}$ and the function that defines the left side of the region is $\underline{\quad y^2 = -4x \quad}$. To find the area, we need to calculate the integral $\int_{-2}^{2} [(\underline{\quad 3 - y^2 \quad}) - (\underline{\quad -\frac{y^2}{4} \quad}$)] dy.

$$\int_{-2}^{2} 3 - y^2 - \left(-\frac{y^2}{4}\right) dy \quad \to \quad \int_{-2}^{2} 3 - y^2 + \frac{y^2}{4}\, dy \quad \to \quad 3y - \frac{1}{3}y^3 + \frac{y^3}{12}\Big|_{-2}^{2}$$

$$3(2)-\frac{1}{3}(2)^3+\frac{2^3}{12}-\left(3(-2)-\frac{1}{3}(-2)^3+\frac{(-2)^3}{12}\right)$$

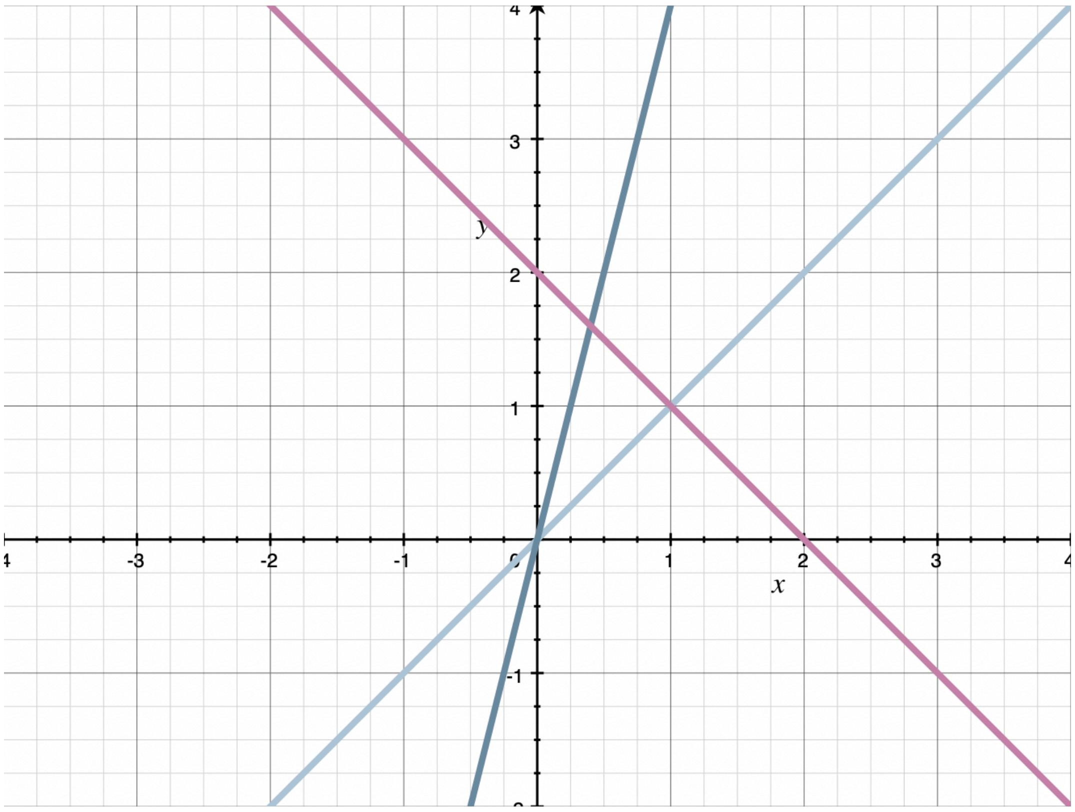
$$6-\frac{8}{3}+\frac{8}{12}+6-\frac{8}{3}+\frac{8}{12} \quad \to \quad 8$$

3. Sketch the graphs of $x^2=y$, $y=0$, and $x=2$. The area of the region they enclose can be found using both functions of x and functions of y. Below is the integral in terms of x. Write the integral in terms of y for the same region, then verify that it produces the same area.

$$\int_0^2 x^2\,dx=\frac{8}{3} \qquad\qquad \int_0^4 2-\sqrt{y}\,dy=\frac{8}{3}$$

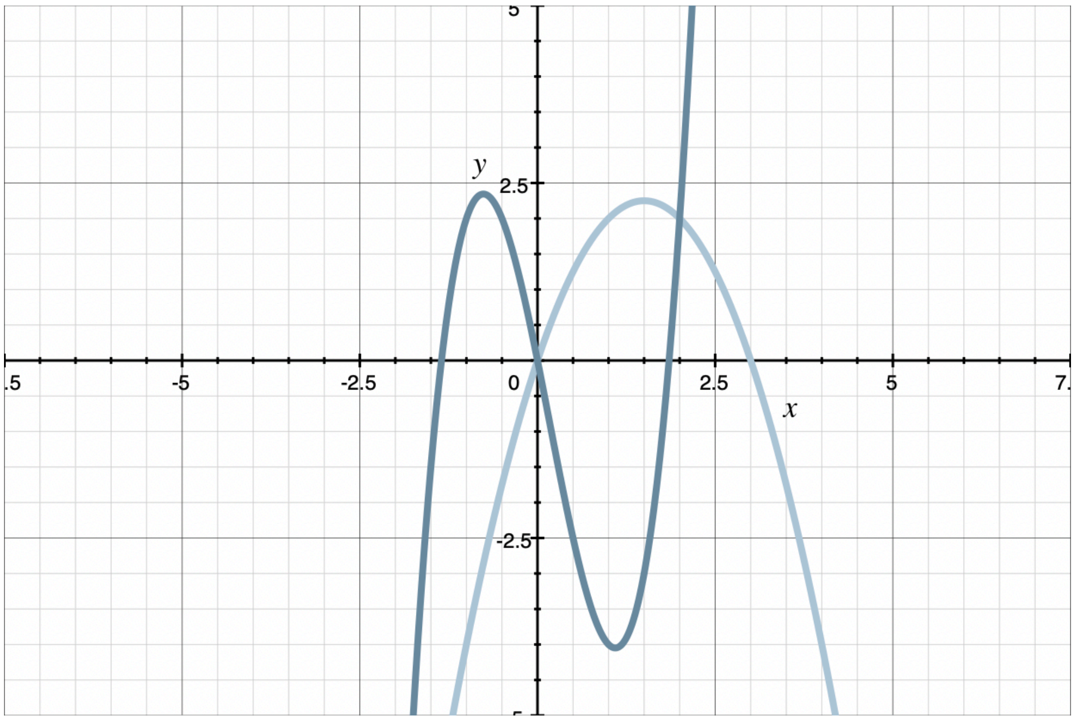
8.6 FINDING THE AREA BETWEEN CURVES THAT INTERSECT AT MORE THAN TWO POINTS

1. Set up and evaluate the integrals that will calculate the area for the region below bounded by $y=x$, $y=4x$, and $y=-x+2$.



$$\int_{0}^{\frac{2}{5}} 4x - x\ dx + \int_{\frac{2}{5}}^{1} -x + 2 - 4x\ dx = \frac{3}{5}$$

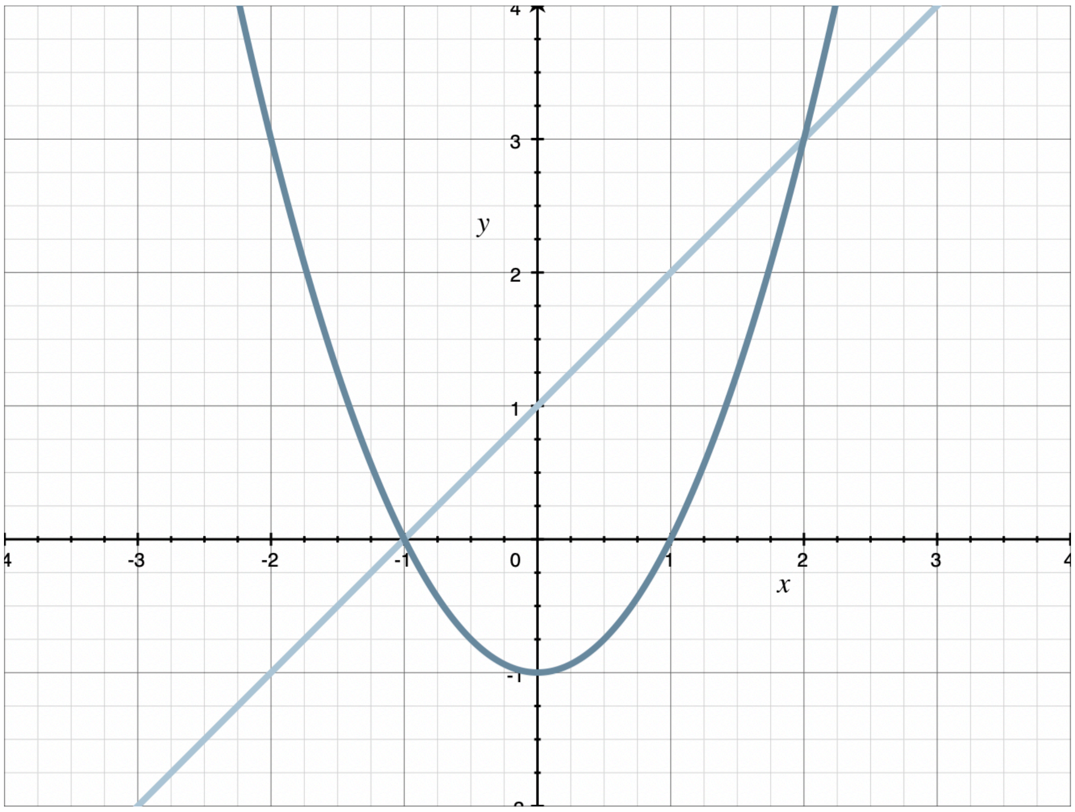
2. The graphs of $y = -x^2 + 3x$ and $y = 2x^3 - x^2 - 5x$ intersect at points $(-2, -10)$, $(0,0)$, and $(2,2)$. Set up the integrals that will find the area of both of the enclosed regions that these functions form.



$$\int_{-2}^{0} (2x^3 - x^2 - 5x) - (-x^2 + 3x)\ dx + \int_{0}^{2} (-x^2 + 3x) - (2x^3 - x^2 - 5x)\ dx$$

8.7 VOLUMES WITH CROSS SECTIONS: SQUARES AND RECTANGLES

1. Set up the integral that would find the volume of the solid whose base is bounded by The base of a solid is bounded by $y = x + 1$ and $y = x^2 - 1$. Set up an integral to find the volume of the solid using cross sections perpendicular to the x-axis,



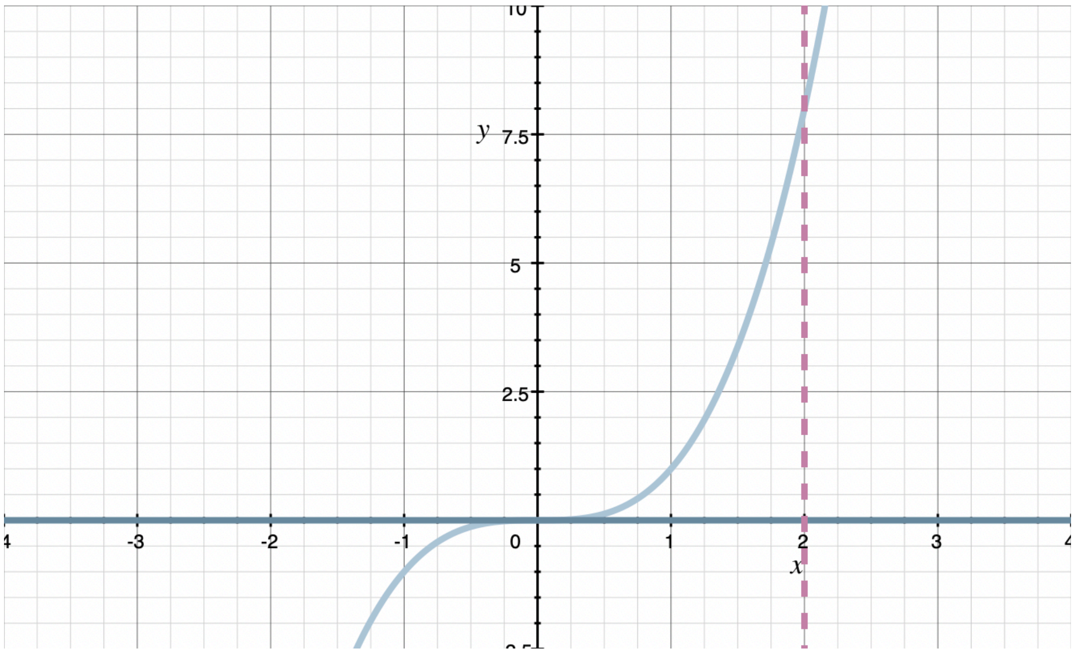
a. when the cross sections are squares.

$$\int_{-1}^{2} (x + 1 - (x^2 - 1))^2\ dx$$

b. when the cross sections are rectangles with height 2.

$$\int_{-1}^{2} 2(x+1-(x^2-1))\ dx$$

2. The graph below shows a region bounded by $y = x^3$, $x = 2$, and $y = 0$, which forms the base of some solid. Match each integral with the statement it represents.



a. The volume given by square cross sections that are perpendicular to the x-axis: $\int_0^2 x^6\ dx$

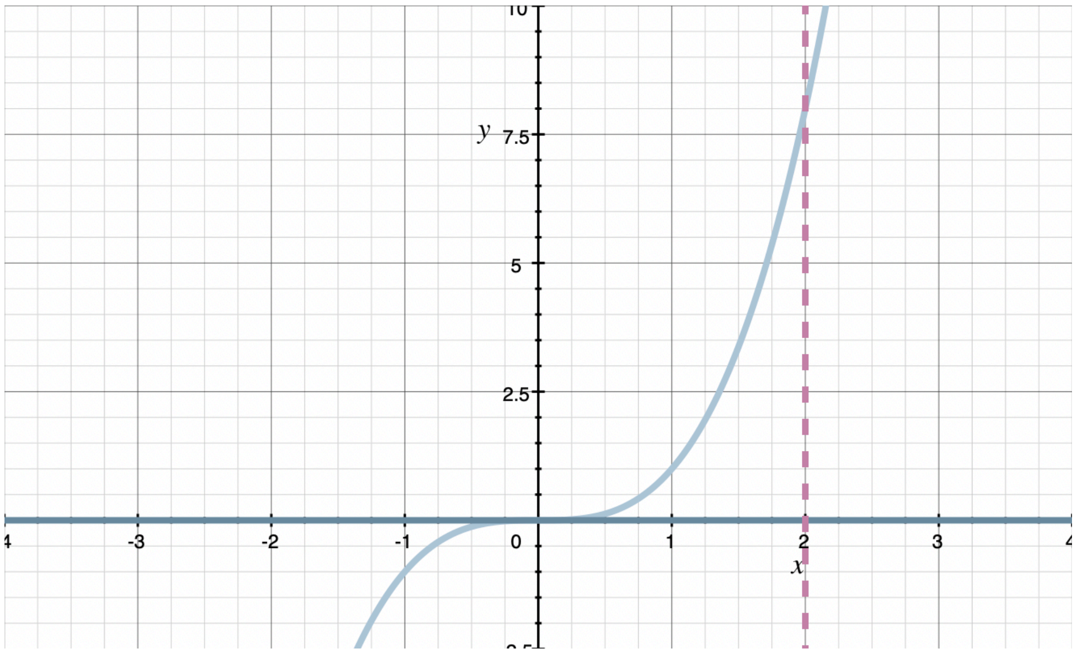
b. The volume given by square cross sections that are perpendicular to the y-axis: $\int_0^8 (2-\sqrt[3]{y})^2\ dy$

c. The volume given by rectangular cross sections that are perpendicular to the x-axis with height 1: $\int_0^2 x^3\ dx$

d. The volume given by rectangular cross sections that are perpendicular to the y-axis with height 1: $\int_0^8 2-\sqrt[3]{y}\ dy$

8.8 VOLUMES WITH CROSS SECTIONS: TRIANGLES AND SEMICIRCLES

1. The graph below shows a region bounded by $y = x^3$, $x = 2$, and $y = 0$, which forms the base of a solid. Match each integral with the statement it represents.



a. Equilateral triangles with the base perpendicular to the x-axis:

$$\frac{\sqrt{3}}{4}\int_0^2 (x^3)^2\ dx = \frac{\sqrt{3}}{4}\int_0^2 x^6\ dx$$

b. Equilateral triangles with the base perpendicular to the y-axis:

$$\frac{\sqrt{3}}{4}\int_0^8 (2 - \sqrt[3]{y})^2\ dy$$

c. Isosceles right triangles with the leg as the base of the solid perpendicular to the x-axis: $\frac{1}{2}\int_0^2 (x^3)^2\ dx = \frac{1}{2}\int_0^2 x^6\ dx$

d. Isosceles right triangles with the leg as the base of the solid perpendicular to the y-axis: $\frac{1}{2}\int_0^8 (2 - \sqrt[3]{y})^2\ dy$

e. Isosceles right triangles with the hypotenuse as the base of the solid perpendicular to the x-axis: $\frac{1}{2}\int_0^2 \left(\frac{x^3}{\sqrt{2}}\right)^2\ dx = \frac{1}{4}\int_0^2 x^6\ dx$

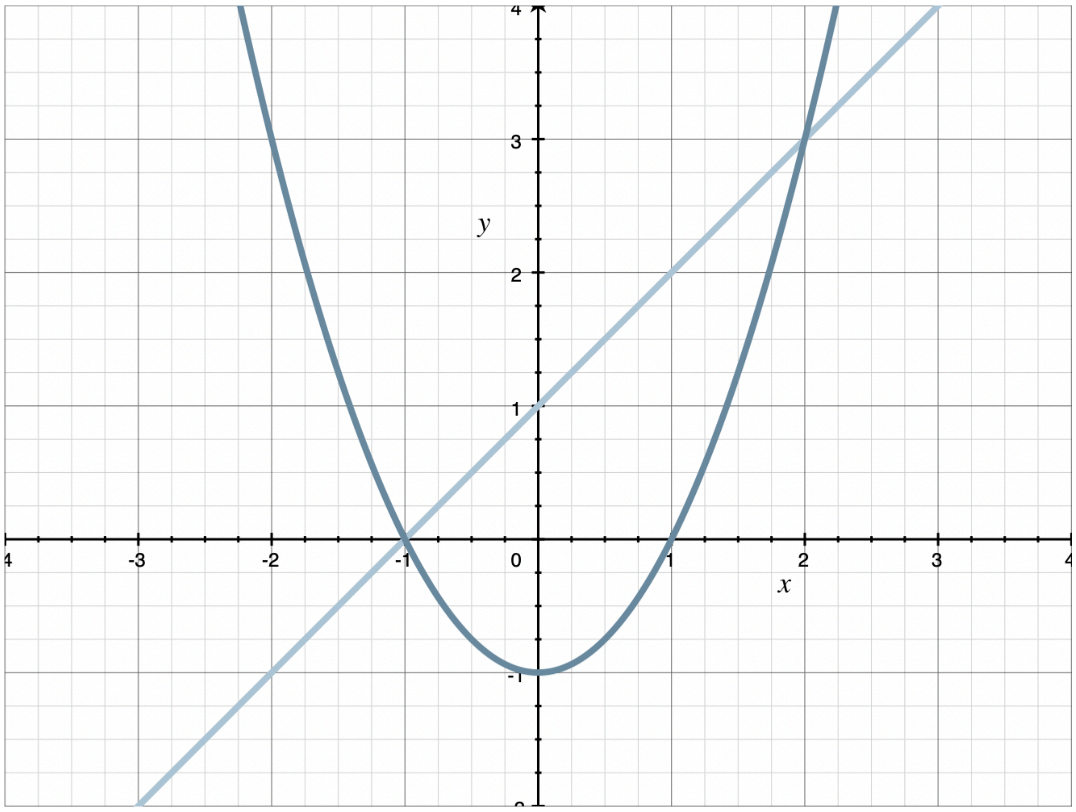
f. Isosceles right triangles with the hypotenuse as the base of the solid perpendicular to the y-axis:

$$\frac{1}{2}\int_0^8 \left(\frac{2-\sqrt[3]{y}}{\sqrt{2}}\right)^2\ dy = \frac{1}{4}\int_0^8 (2-\sqrt[3]{y})^2\ dy$$

g. Semicircles with the the diameter as the base perpendicular to the x-axis: $\frac{\pi}{2}\int_0^2 \left(\frac{x^3}{2}\right)^2\ dx = \frac{\pi}{8}\int_0^2 x^6\ dx$

h. Semicircles with the the diameter as the base perpendicular to the y-axis: $\frac{\pi}{2}\int_0^8 \left(\frac{2-\sqrt[3]{y}}{2}\right)^2\ dy = \frac{\pi}{8}\int_0^8 (2-\sqrt[3]{y})^2\ dy$

2. Set up the integral that calculates the volume of the solid whose base is bounded by $y = x + 1$ and $y = x^2 - 1$, using cross sections that are perpendicular to the x-axis that are:



a. Equilateral triangles:

$$\frac{\sqrt{3}}{4}\int_{-1}^{2}(x+1-(x^2-1))^2\ dx=\frac{\sqrt{3}}{4}\int_{-1}^{2}(-x^2+x+2)^2\ dx$$

b. Isosceles right triangles with the leg as the base of the solid:

$$\frac{1}{2}\int_{-1}^{2}(x+1-(x^2-1))^2\ dx=-\frac{1}{2}\int_{1}^{2}(-x^2+x+2)^2\ dx$$

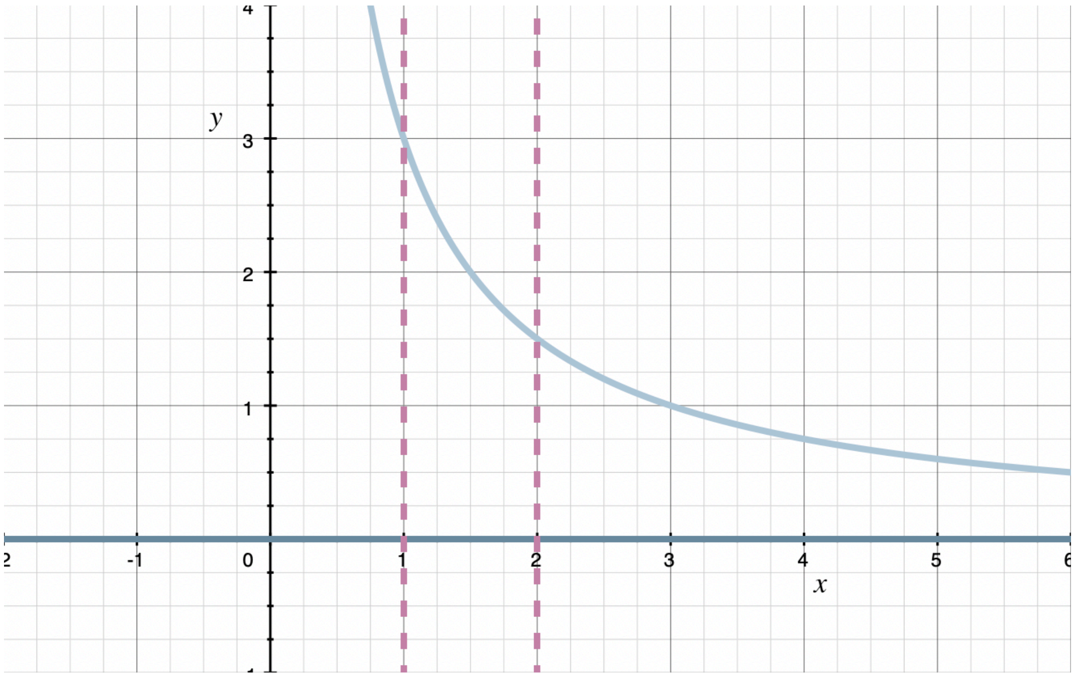
c. Isosceles right triangles with the hypotenuse as the base of the solid: $\frac{1}{2}\int_{-1}^{2}\left(\frac{x+1-(x^2-1)}{\sqrt{2}}\right)^2\ dx=\frac{1}{4}\int_{-1}^{2}(-x^2+x+2)^2\ dx$

d. Semicircles with the diameter as the base:

$$\frac{\pi}{2}\int_{-1}^{2}\left(\frac{x+1-(x^2-1)}{2}\right)^2\ dx=\frac{\pi}{8}\int_{-1}^{2}(-x^2+x+2)^2\ dx$$

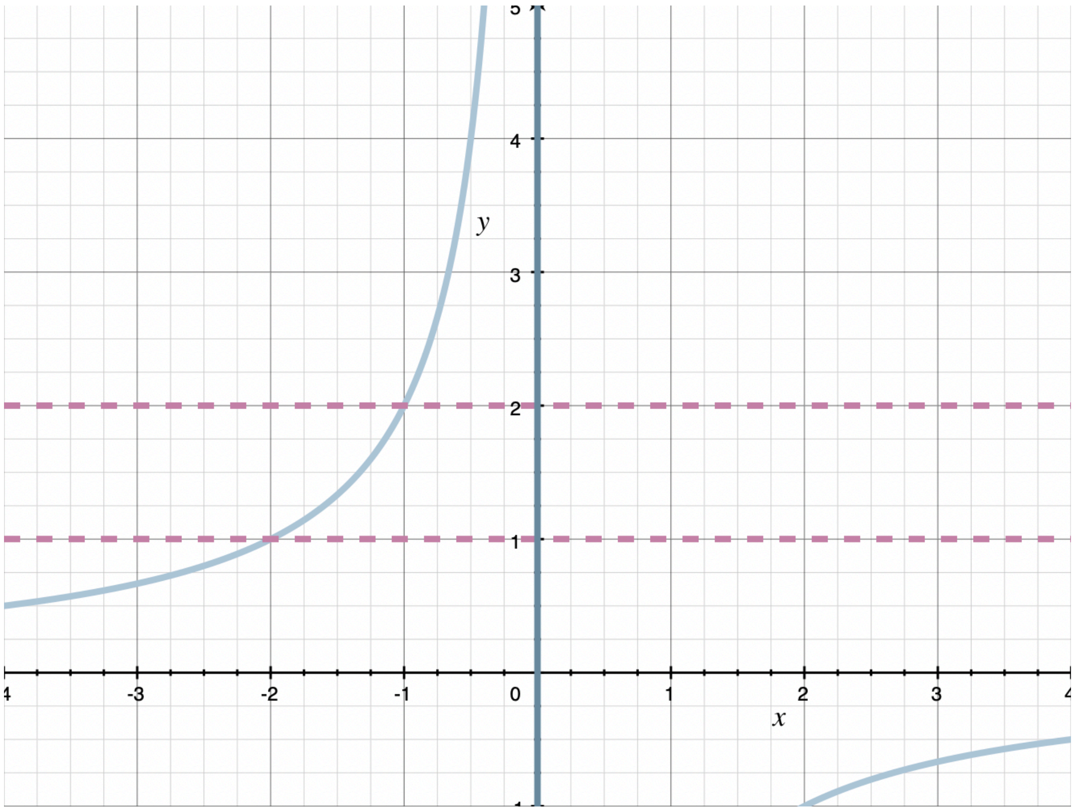
8.9 VOLUME WITH DISC METHOD: REVOLVING AROUND THE X- OR Y-AXIS

1. The graph shows the region enclosed by $g(x) = \frac{3}{x}$, $x = 1$, $x = 2$, and $y = 0$. Set up an integral that would find the volume of this region rotated around the x-axis.



$$\pi \int_0^2 \left(\frac{3}{x}\right)^2 dx$$

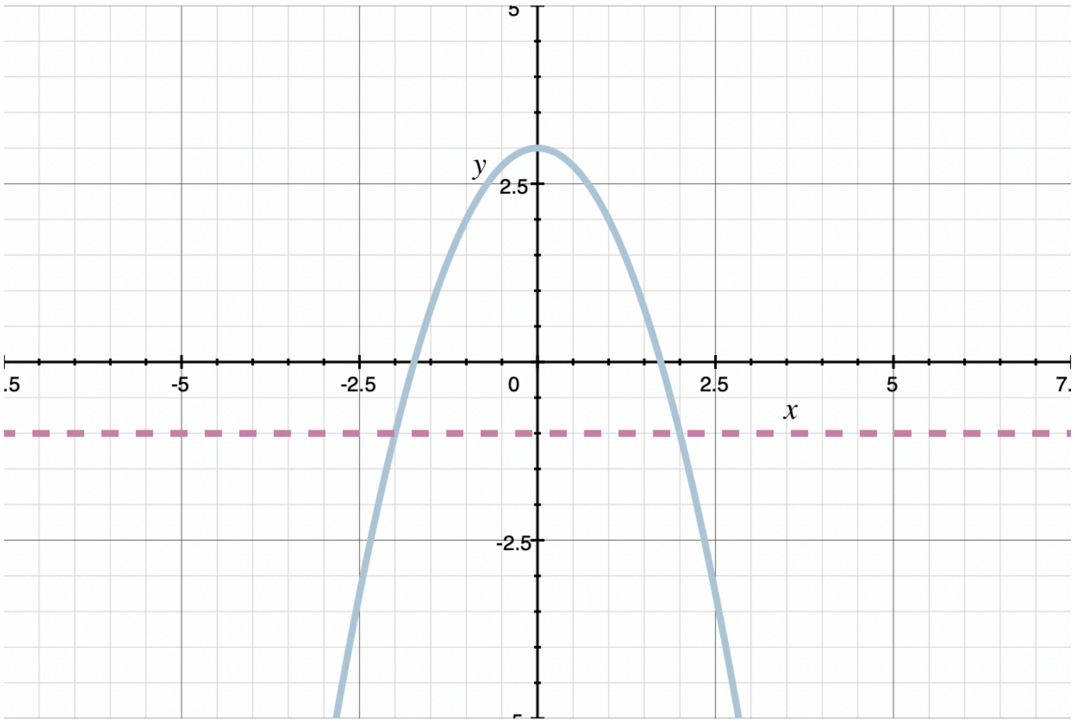
2. The graph shows the region enclosed by $x = -\frac{2}{y}$, $y = 1$, $y = 2$, and $x = 0$. Set up an integral that would find the volume of this region rotated around the y-axis.



$$\pi\int_1^2\left(-\frac{2}{y}\right)^2\ dy$$

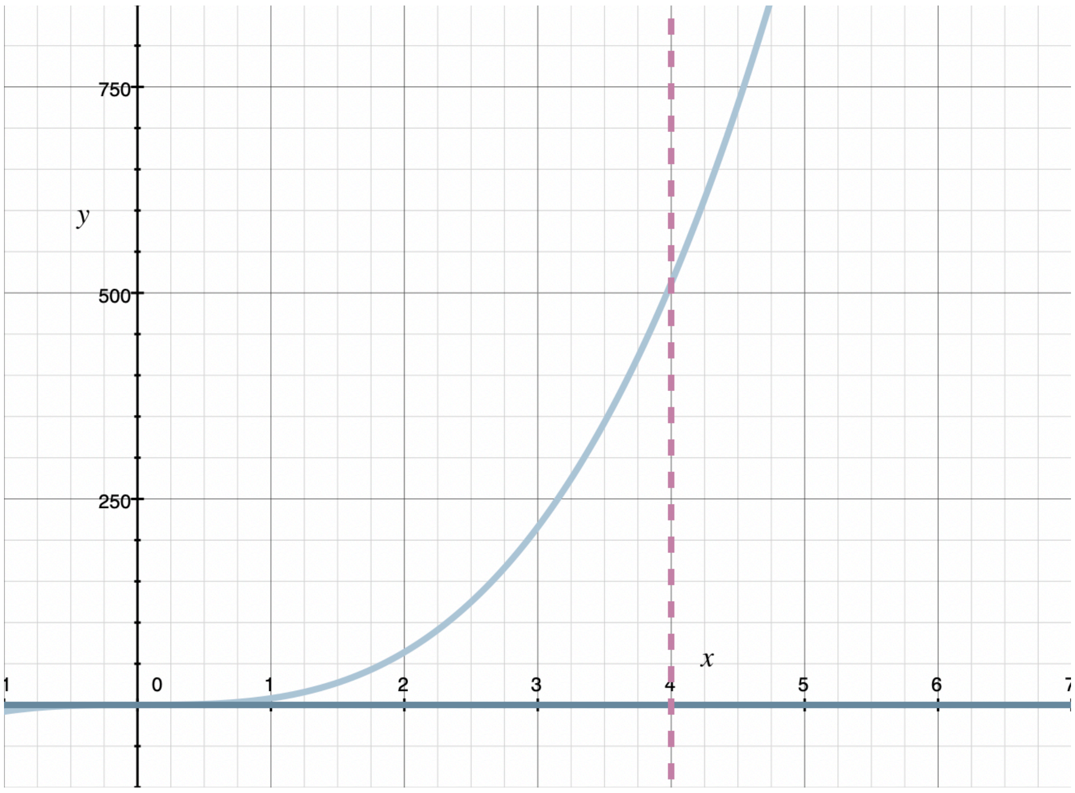
8.10 VOLUME WITH DISC METHOD: REVOLVING AROUND OTHER AXES

1. Sketch the graphs of $f(x)=3-x^2$ and $y=-1$, then set up an integral to find the volume when this region is rotated around $y=-1$.



$$\pi\int_{-2}^{2}(3-x^2-(-1))^2\ dx=\pi\int_{-2}^{2}(3-x^2+1)^2\ dx$$

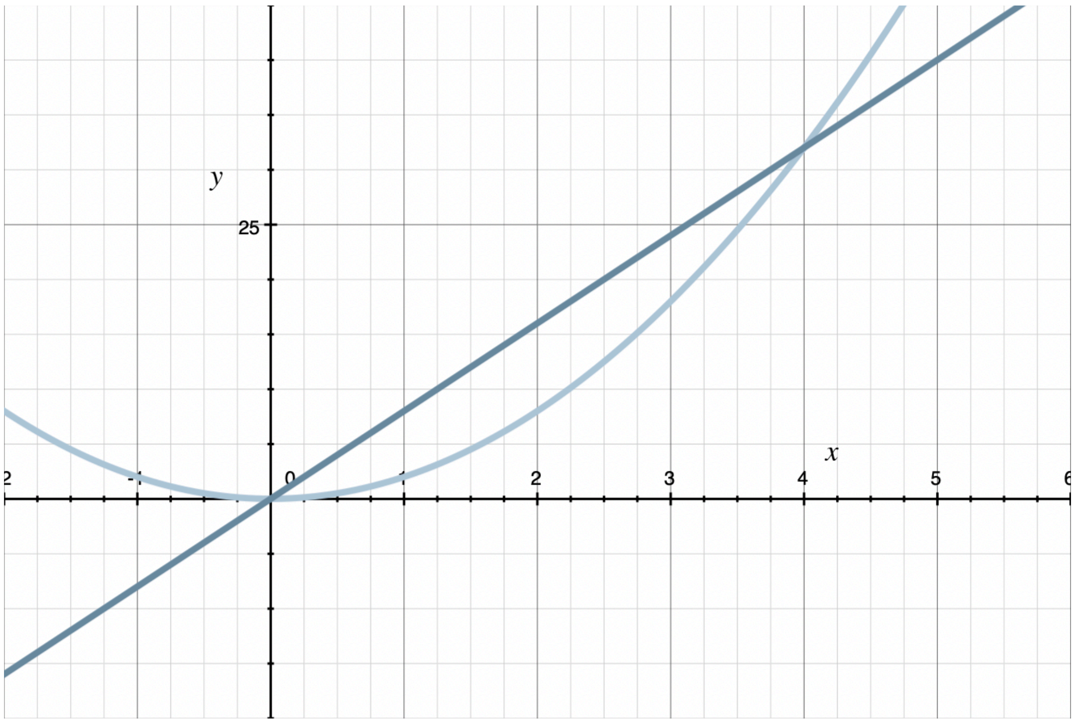
2. The graph shows the shaded region enclosed by $\frac{1}{2}y^{\frac{1}{3}}=x$, $x=4$, and $y=0$. Set up an integral to find the volume when the region is rotated around $x=4$.



$$\pi\int_{0}^{512}\left(4-\frac{1}{2}y^{\frac{1}{3}}\right)^{2}\ dy$$

8.11 VOLUME WITH WASHER METHOD, REVOLVING AROUND THE X- OR Y-AXIS

1. The graph below shows the region enclosed by $y = 2x^2$ and $y = 8x$.



a. Set up the integral that would calculate the volume of the region when rotated around the x-axis.

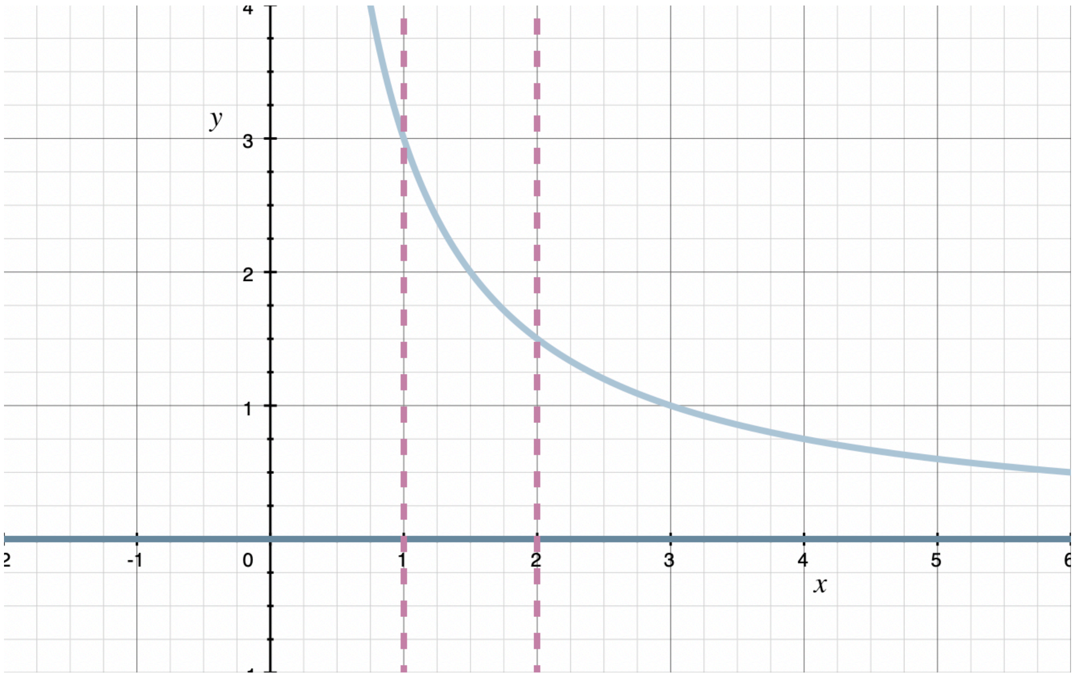
$$\pi\int_{0}^{4}(8x-0)^2-(2x^2-0)^2\ dx=\pi\int_{0}^{4}(8x)^2-(2x^2)^2\ dx$$

b. Set up the integral that would calculate the volume of the region when rotated around the y-axis.

$$\pi\int_0^{32}\left(\sqrt{\frac{y}{2}}\right)^2-\left(\frac{y}{8}\right)^2\ dy$$

8.12 VOLUME WITH WASHER METHOD: REVOLVING AROUND OTHER AXES

1. The graph shows the region enclosed by $g(x)=\frac{3}{x}$, $x=1$, $x=2$, and $y=0$.



Set up an integral to calculate the volume when the region is rotated around,

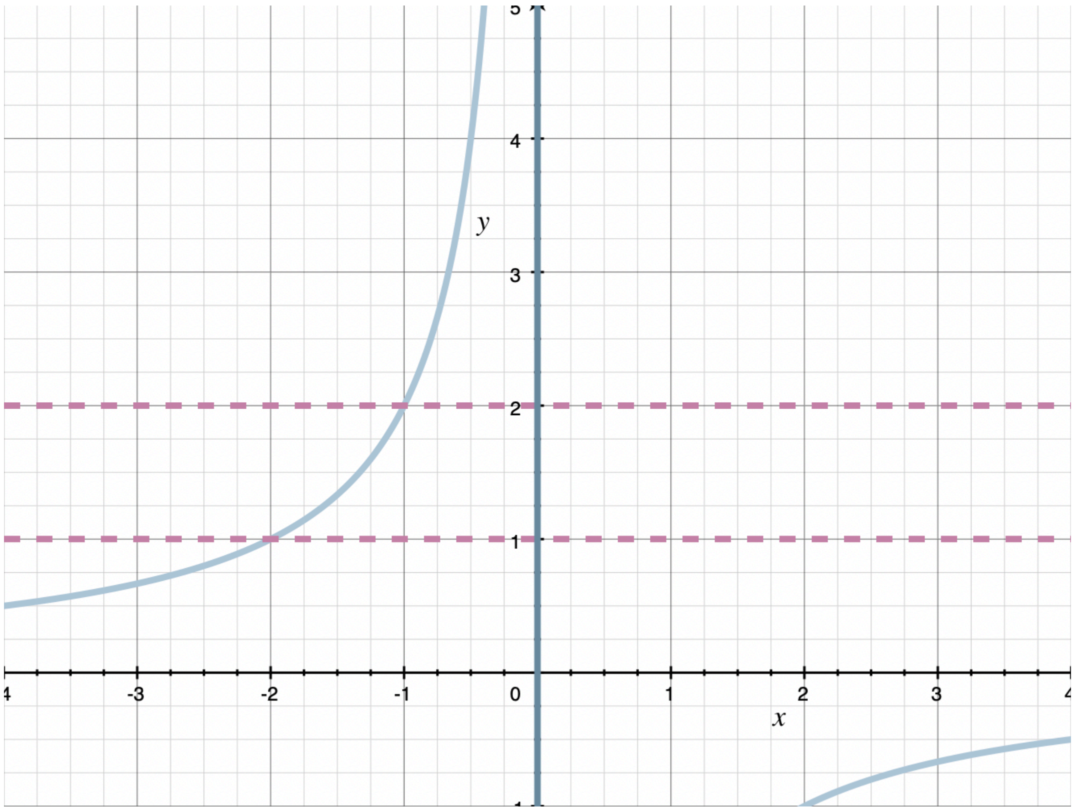
a. the line $y=-2$:

$$\pi\int_0^{2}\left(\frac{3}{x}-(-2)\right)^2-(0-(-2))^2\ dx=\pi\int_0^{2}\left(\frac{3}{x}+2\right)^2-2^2\ dx$$

b. the line $y=6$:

$$\pi\int_0^2 (6-0)^2-\left(6-\frac{3}{x}\right)^2\ dx=\pi\int_0^2 6^2-\left(6-\frac{3}{x}\right)^2\ dx$$

2. The graph shows the region enclosed by $x=-\frac{2}{y}$, $y=1$, $y=2$, and $x=0$.



Set up an integral that would calculate the volume of the region rotated around,

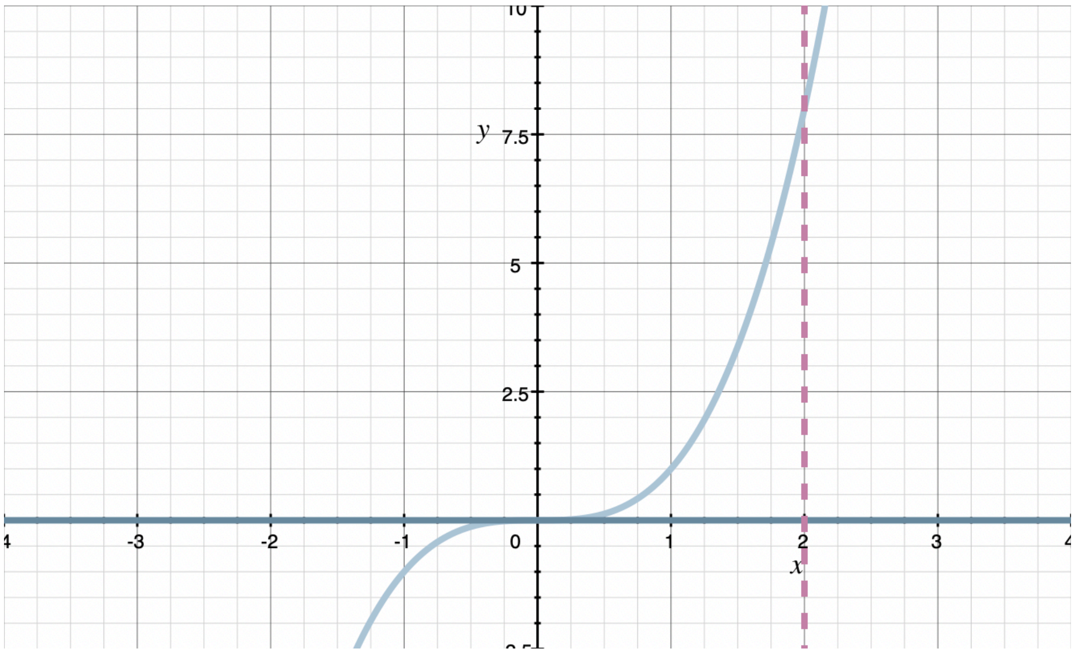
a. the line $x=2$:

$$\pi\int_1^2 \left(2-\left(-\frac{2}{y}\right)\right)^2-(2-0)^2\ dy=\pi\int_1^2 \left(2+\frac{2}{y}\right)^2-2^2\ dy$$

b. the line $x=-4$:

$$\pi\int_1^2 (0-(-4))^2-\left(-\frac{2}{y}-(-4)\right)^2\ dy=\pi\int_1^2 4^2-\left(-\frac{2}{y}+4\right)^2\ dy$$

3. The graph shows the base of a solid, which is bounded by $y = x^3$, $x = 2$, and $y = 0$. Explain what each integral represents.



a. $\pi \int_0^2 (x^3 - (-4))^2 - (0 - (-4))^2 \, dx$: This integral represents the volume of the region when rotated around $y = -4$.

b. $\pi \int (5 - \sqrt[3]{y})^2 - (5 - 2)^2 \, dy$: This integral represents the volume of the region when rotated around $x = 5$.