■ 1. Given $x(t) = t^3 - 3t^2 + 4$ and $y(t) = 6t^2 - 2t$, find $\frac{d^2y}{dx^2}$ when $t = 3$.

A $\frac{24}{9}$

B $-\frac{20}{9}$

$\boxed{\text{C}}$ $-\frac{100}{243}$

D $-\frac{24}{9}$

Solution: C

Find the first derivative of the parametric curve.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{12t - 2}{3t^2 - 6t}$$

Use the first derivative, along with quotient rule, to find the second derivative of the parametric curve.

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dx}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{(12)(3t^2 - 6t) - (12t - 2)(6t - 6)}{(3t^2 - 6t)^2}}{3t^2 - 6t}$$

$$= \frac{36t^2 - 72t - 72t^2 + 72t + 12t - 12}{(3t^2 - 6t)^3} = \frac{-12(3t^2 - t + 1)}{(3t^2 - 6t)^3}$$

Evaluate the second derivative at $t = 3$.

$$\frac{d^2y}{dx^2}(3) = \frac{-12(3(3)^2 - 3 + 1)}{(3(3)^2 - 6(3))^3} = \frac{-12(27 - 3 + 1)}{(27 - 18)^3} = -\frac{100}{243}$$

■ 2. Write the equation of the tangent line at $t=\frac{\pi}{3}$ for the parametric equation $x(t)=-\cos t$ and $y(t)=\sin t$.

$\boxed{\text{A}}$ $y-\frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{3}\left(x+\frac{1}{2}\right)$

B $y-\frac{1}{2}=\sqrt{3}\left(x+\frac{\sqrt{3}}{2}\right)$

C $y+\frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{3}\left(x-\frac{1}{2}\right)$

D $y+\frac{1}{2}=\sqrt{3}\left(x-\frac{\sqrt{3}}{2}\right)$

Solution: A

Evaluate x and y at $t=\frac{\pi}{3}$.

$$x\left(\frac{\pi}{3}\right)=-\cos\left(\frac{\pi}{3}\right)=-\frac{1}{2}$$

$$y\left(\frac{\pi}{3}\right)=\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$$

Find the slope of the tangent line. With $x'(t)=\sin t$ and $y'(t)=\cos t$, then the first derivative is

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\cos t}{\sin t}$$

$$\frac{dy}{dx}\left(\frac{\pi}{3}\right)=\frac{\cos\left(\frac{\pi}{3}\right)}{\sin\left(\frac{\pi}{3}\right)}=\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}=\frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$$

Finally, write the equation of the tangent line.

$$y-\frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{3}\left(x+\frac{1}{2}\right)$$

■ 3. Which of the following gives the length of the curve defined by the parametric equations $x(t)=\frac{2}{3}t^3$ and $y(t)=t^2$ from $t=1$ to $t=3$?

A $\int_1^3 \sqrt{1+2t}\ dt$

B $\int_1^3 \sqrt{1+4t^2}\ dt$

C $\int_1^3 \sqrt{2t^2+2t}\ dt$

[✓ D] $\int_1^3 \sqrt{4t^4+4t^2}\ dt$

Solution: D

When we plug the derivatives $x'(t)=2t^2$ and $y(t)=2t$, and the given interval, into the formula for arc length we get

$$L=\int_\alpha^\beta \sqrt{(x'(t))^2+(y'(t))^2}\ dt$$

$$L=\int_1^3 \sqrt{(2t^2)^2+(2t)^2}\ dt$$

$$L=\int_1^3 \sqrt{4t^4+4t^2}\ dt$$

■ 4. Find the velocity vector given the position vector $\langle t\ln t, \ln t^2\rangle$.

A $\left\langle 1, \frac{1}{t^2} \right\rangle$

$\boxed{\text{B}}$ $\left\langle 1+\ln t, \frac{2}{t} \right\rangle$

C $\left\langle 1, \frac{2}{t} \right\rangle$

D $\left\langle \ln t, \frac{1}{t^2} \right\rangle$

Solution: B

The velocity vector is represented by $\langle x'(t), y'(t) \rangle$, so we'll take the derivatives of each component of the position vector.

$$\left\langle t\left(\frac{1}{t}\right)+(1)\ln t, \frac{1}{t^2}(2t) \right\rangle$$

$$\left\langle 1+\ln t, \frac{2t}{t} \right\rangle$$

■ 5. The position of a particle moving in the xy-plane is given by the parametric curve $f(x) = \langle x(t), y(t) \rangle$, where $\frac{dx}{dt} = t^2e^t$ and $\frac{dy}{dt} = \frac{5}{3}t^2$. If $f(1) = \left\langle e-2, \frac{5}{9} \right\rangle$, use a calculator to find $f(3)$.

A $\langle 180.770, 15 \rangle$

$\boxed{\text{B}}$ $\langle 98.428, 15 \rangle$

C 99.061

D 181.391

Solution: B

To find position at a specific time, the initial condition must be added to the integral over the interval that spans the starting time to the ending time.

$$\left\langle e-2+\int_1^3 t^2e^t\ dt, \frac{5}{9}+\int_1^3 \frac{5}{3}t^2\right\rangle$$

Use a calculator to find $\langle 98.428, 15 \rangle$.

■ 6. For time $t > 0$, the position of an object moving in the xy-plane is given by the parametric equation $x(t) = \cos 2t$ and $y(t) = -3t^2 + 7t$. Which of the following represents the speed of the object?

A $\int_0^1 \sqrt{(\cos 2t)^2 + (-3t^2 + 7t)^2}\ dt$

B $\sqrt{(\cos 2t)^2 + (-3t^2 + 7t)^2}$

C $\int_0^1 \sqrt{(-2\sin 2t)^2 + (-6t + 7)^2}\ dt$

$\boxed{\text{D}}$ $\sqrt{(-2\sin 2t)^2 + (-6t + 7)^2}$

Solution: D

Substituting the derivatives $x'(t) = -2\sin 2t$ and $y'(t) = -6t + 7$ into the formula for speed of a parametric function is given by

$$\sqrt{(x'(t))^2 + (y'(t))^2}$$

$$\sqrt{(-2\sin 2t)^2+(-6t+7)^2}$$

■ 7. What is the slope of the line tangent to the polar curve $r=\sin(2\theta)$ when $\theta=\frac{2\pi}{3}$?

A $-\frac{3\sqrt{3}}{5}$

B 1

$\boxed{\text{C}}$ $\frac{3\sqrt{3}}{5}$

D -1

Solution: C

With $y=r\sin\theta$ and $x=r\cos\theta$, we can say

$$y=r\sin\theta=\sin(2\theta)\sin\theta$$

$$x=r\cos\theta=\sin(2\theta)\cos\theta$$

Then the slope of the tangent line is

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{2\cos(2\theta)\sin\theta+\sin(2\theta)\cos\theta}{2\cos(2\theta)\cos\theta-\sin(2\theta)\sin\theta}$$

At $\theta=\frac{2\pi}{3}$, the slope is

$$\frac{dy}{dx}=\frac{2\cos\left(2\cdot\frac{2\pi}{3}\right)\sin\frac{2\pi}{3}+\sin\left(2\cdot\frac{2\pi}{3}\right)\cos\frac{2\pi}{3}}{2\cos\left(2\cdot\frac{2\pi}{3}\right)\cos\frac{2\pi}{3}-\sin\left(2\cdot\frac{2\pi}{3}\right)\sin\frac{2\pi}{3}}$$

$$\frac{dy}{dx}=\frac{2\cos\frac{4\pi}{3}\sin\frac{2\pi}{3}+\sin\frac{4\pi}{3}\cos\frac{2\pi}{3}}{2\cos\frac{4\pi}{3}\cos\frac{2\pi}{3}-\sin\frac{4\pi}{3}\sin\frac{2\pi}{3}}=\frac{2\left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)-\frac{\sqrt{3}}{2}\left(-\frac{1}{2}\right)}{2\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)-\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}$$

$$\frac{dy}{dx}=\frac{-\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{4}}{\frac{1}{2}+\frac{3}{4}}=\frac{-\frac{\sqrt{3}}{4}}{\frac{5}{4}}=-\frac{\sqrt{3}}{5}$$

■ 8. Use a calculator to find any points on the interval $0 \le \theta < \frac{\pi}{2}$ at which the slope of the tangent line to $r = 2 - 3\sin\theta$ is horizontal.

I. $\theta = 0.340$

II. $\theta = 0.730$

III. $\theta = 1.571$

$\boxed{\text{A}}$	I only	B	II only
C	I and III	D	II and III

Solution: A

Horizontal tangent lines exist where $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$.

$$y = r\sin\theta$$

$$y = (2 - 3\sin\theta)\sin\theta$$

$$y = 2\sin\theta - 3\sin^2\theta$$

$$\frac{dy}{d\theta} = 2\cos\theta - 6\sin\theta\cos\theta$$

So horizontal tangent lines exist at

$$2\cos\theta - 6\sin\theta\cos\theta = 0$$

$$2\cos\theta(1 - 3\sin\theta) = 0$$

$$\theta = 0.340$$

■ 9. Which integral represents the area of the inner loop of $r = 1 + 2\cos\theta$.

A $\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (1 + 2\cos\theta)^2 \ d\theta$

B $\int_{0}^{\pi} (1 + 2\cos\theta)^2 \ d\theta$

$\boxed{\text{C}}$ $\frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (1 + 2\cos\theta)^2 \ d\theta$

D $\frac{1}{2}\int_{0}^{\pi} (1 + 2\cos\theta)^2 \ d\theta$

Solution: C

Solve $r = 0$ to find the bounds for the integral.

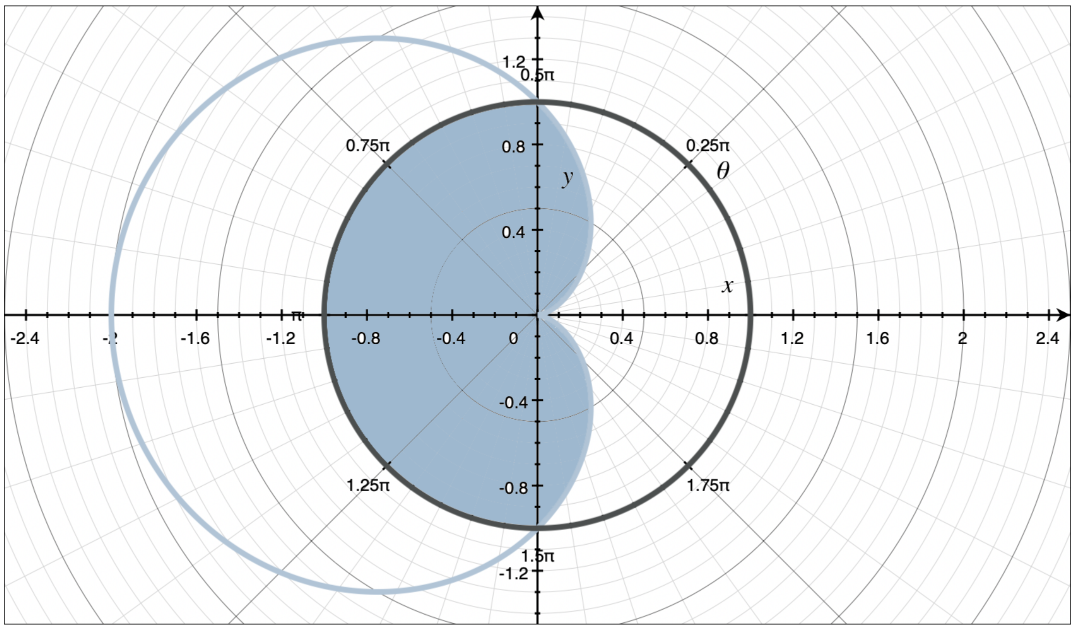
$$1 + 2\cos\theta = 0$$

$$\cos\theta = -\frac{1}{2}$$

$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

So the area bounded by this part of the polar curve is

$$A=\frac{1}{2}\int_{\alpha}^{\beta}(r(\theta))^2\ d\theta=\frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}(1+2\cos\theta)^2\ d\theta$$



■ **10. Find the area enclosed by $r=1-\cos\theta$ and $r=1$ as shown in the graph above.**

A $\frac{3\pi}{4}-2$

$\boxed{\text{B}}$ $\frac{5\pi}{4}-2$

C $\frac{3\pi}{4}+2$

D $\frac{5\pi}{4}+2$

Solution: B

The area contained between the two curves consists of the left half of the circle $r=1$, plus the area inside $r=1-\cos\theta$ to the right of the vertical axis. The area of the half circle is $\frac{\pi}{2}$, so total area is

$$A=\frac{\pi}{2}+2\left(\frac{1}{2}\right)\int_{0}^{\frac{\pi}{2}}(1-\cos\theta)^2\ d\theta$$

$$A=\frac{\pi}{2}+\int_0^{\frac{\pi}{2}}1-2\cos\theta+\cos^2\theta\ d\theta$$

Use the power-reducing formula for $\cos^2\theta$.

$$A=\frac{\pi}{2}+\int_0^{\frac{\pi}{2}}1-2\cos\theta+\frac{1+\cos 2\theta}{2}\ d\theta$$

$$A=\frac{\pi}{2}+\left(\theta-2\sin\theta+\frac{1}{2}\theta+\frac{\sin 2\theta}{4}\right)\Big|_0^{\frac{\pi}{2}}$$

$$A=\frac{\pi}{2}+\frac{\pi}{2}-2\sin\frac{\pi}{2}+\frac{1}{2}\left(\frac{\pi}{2}\right)+\frac{\sin\left(2\frac{\pi}{2}\right)}{4}-\left(0-2\sin(0)+\frac{1}{2}(0)+\frac{\sin(2(0))}{4}\right)$$

$$A=\frac{\pi}{2}+\frac{\pi}{2}-2(1)+\frac{\pi}{4}+\frac{\sin\pi}{4}$$

$$A=\frac{5\pi}{4}-2$$

■ 11. A particle moves along the xy-plane, and the position of the particle at time t, measured in seconds, is represented by $(x(t), y(t))$, measured in centimeters. The particle starts at the position $(0,0)$ at $t=0$, and $x'(t)=2.5\cos(3t)$ and $y'(t)=4\sin(1.5t)$.

a. Find the speed of the particle at time $t=4$ seconds. Indicate the units of measure.

b. At time $t=1.5$ seconds, use a calculator to determine whether the speed of the particle is increasing or decreasing. Explain your answer.

c. Use a calculator to find the total distance the particle traveled from $t = 1$ to $t = 4$ seconds.

d. Find the x-coordinate of the position of the particle at time $t = 2$ seconds.

Solution:

a. The speed is given by

$$s(t) = \sqrt{(x'(t))^2 + (y'(t))^2}$$

$$s(t) = \sqrt{(2.5\cos(3t))^2 + (4\sin(1.5t))^2}$$

$$s(t) = \sqrt{6.25\cos^2(3t) + 16\sin^2(1.5t)}$$

At $t = 4$,

$$s(t) = \sqrt{6.25\cos^2(12) + 16\sin^2(6)} \approx 2.481 \text{ cm/sec}$$

b. Use a calculator to find $s'(1.5) \approx -4.940$. Since $s'(1.5) < 0$, the speed of the particle is decreasing at $t = 1.5$ seconds.

c. Use the arc length formula and a calculator to find

$$L = \int_{\alpha}^{\beta} \sqrt{(x'(t))^2 + (y'(t))^2}\ dt$$

$$L = \int_{1}^{4} \sqrt{(2.5\cos(3t))^2 + (4\sin(1.5t))^2}\ dt \approx 9.867$$

d. To find the position value at a specific time, the initial condition must be added to an integral with bounds from the start time to the end time. Use a calculator to evaluate the integral.

$$x(2) = 0 + \int_0^2 x'(t)\ dt = \int_0^2 2.5\cos(3t)\ dt \approx -0.233$$