9.1 DEFINING AND DIFFERENTIATING PARAMETRIC EQUATIONS

1. If $x(t) = t^2 + 1$, then $\frac{dx}{dt} =$ $\underline{\quad 2t \quad}$. If $y(t) = \frac{2}{3}t^{\frac{3}{2}}$, then $\frac{dy}{dt} =$ $\underline{\quad \sqrt{t} \quad}$.

   This means $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} =$ $\underline{\quad \frac{\sqrt{t}}{2t} \quad}$.

2. A particle moves along a curve in the xy-plane with position $((x(t), y(t))$. Complete the following table.

t	0	2	4	6
dy/dt	3	-1	0	4
dx/dt	2	-3	5	-1/2
dy/dx	3/2	1/3	0	-8

3. Circle the equation of the tangent line for $t = 4$ when $x(t) = 2t + 1$ and $y(t) = \sqrt{t}$ for $t > 0$.

$y - 9 = \frac{1}{8}(x - 2)$

$y - 2 = 8(x - 4)$

$y - 2 = \frac{1}{8}(x - 4)$

$y - 2 = \frac{1}{8}(x - 9)$

9.2 SECOND DERIVATIVES OF PARAMETRIC EQUATIONS

1. If $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 4t$, find the values below.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \qquad \frac{4t}{2} = 2t$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \qquad \frac{d}{dt}(2t) = 2$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \qquad \frac{\frac{d}{dt}(2t)}{2} = \frac{2}{2} = 1$$

2. The position of a particle moving in the xy-plane is given by $x(t) = t^2 - 5$ and $y(t) = t^3$. Find each of the following values.

$$\frac{dx}{dt} = \qquad \frac{dx}{dt} = 2t$$

$$\frac{dy}{dt} = \qquad \frac{dy}{dt} = 3t^2$$

$$\frac{dy}{dx} = \qquad \frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3t}{2}$$

$$\frac{d^2y}{dx^2} = \qquad \frac{d^2y}{dx^2} = \frac{3}{4t}$$

3. For what values of t over the interval $[0,2\pi)$ is $x = \cos t$ and $y = \sin t$ concave up? Explain your answer.

$\frac{dx}{dt} = -\sin t$ and $\frac{dy}{dt} = \cos t$, so $\frac{dy}{dx} = -\frac{\cos t}{\sin t}$

$$\frac{d^2y}{dx^2} = -\frac{1}{\sin^3 t}$$

The second derivative will be positive wherever $\sin t$ is negative. On the interval $[0,2\pi)$, the sine function is negative on $(\pi,2\pi)$, so the parametric curve is concave up on $(\pi,2\pi)$.

9.3 FINDING ARC LENGTHS OF CURVES GIVEN BY PARAMETRIC EQUATIONS

1. Circle the integral below that will give the length of the curve given by $x(t)=\frac{2}{3}t^3$ and $y(t)=\frac{1}{2}t^2$ from $t=0$ to $t=2$.

$$\int_0^1 \sqrt{4t^4+t^2}\ dt \qquad \int_0^1 \sqrt{2t^2+t}\ dt$$

$$\int_0^1 2t^2+t\ dt \qquad \int_0^1 \sqrt{\frac{4}{9}t^6+\frac{1}{4}t^4}\ dt$$

2. Set up and evaluate the integral that will find the length of the curve of $x(t)=\cos t$ and $y(t)=\sin t$ from $t=0$ to $t=2\pi$.

$$\int_0^{2\pi} \sqrt{(-\sin t)^2+(\cos t)^2}\ dt = \int_0^{2\pi} \sqrt{1}\ dt = \int_0^{2\pi} 1\ dt = 2\pi$$

9.4 DEFINING AND DIFFERENTIATING VECTOR-VALUED FUNCTIONS

1. A particle moves in the plane such that its position at any time $t \geq 0$ is given by $h(t)=\langle 3t^2, 2\cos t\rangle$. Find each of the following values.

$h'(t) = \langle 6t, -2\sin t\rangle$

$h''(t) = \langle 6, -2\cos t\rangle$

$h(0) = \langle 0,2\rangle$

$h'(0) = \langle 0,0\rangle$

$h''(0) = \langle 6, -2\rangle$

2. A particle moves in the plane so that its position at any time $t \geq 0$ is given by $h(t) = \langle e^t - t, e^t - e^3 t\rangle$. Choose the statement(s) below that are true about the particle.

   I. At $t = 0$, the particle is at rest.

   II. At $t = 3$, the particle is at rest.

   III. The particle is never at rest.

Statement III is true. At $t = 0$, the x-component of velocity is zero, and at $t = 3$, the y-component of velocity is zero, but for the particle to be at rest, both the x- and y-components have to be zero at the same time.

9.5 INTEGRATING VECTOR-VALUED FUNCTIONS

1. Fill in the blanks from the answer bank below to form true statements.

$$x(t_2) = \underline{x(t_1)} + \underline{\int_{t_1}^{t_2} x'(t)\ dt}$$

$$y(t_1) = \underline{y(t_2)} - \underline{\int_{t_1}^{t_2} y'(t)\ dt}$$

$$\underline{x'(t_2)} = x'(t_1) + \underline{\int_{t_1}^{t_2} x''(t)\ dt}$$

$$\int_{t_1}^{t_2} y''(t)\ dt = \underline{y'(t_2)} - \underline{y'(t_1)}$$

Answer bank:

$x(t_1)$	$\int_{t_1}^{t_2} x'(t)\ dt$	$\int_{t_1}^{t_2} y'(t)\ dt$	$y'(t_1)$
$x'(t_2)$	$\int_{t_1}^{t_2} x''(t)\ dt$	$y'(t_2)$	$y(t_2)$

2. An object moving along a curve in the xy-plane has position $(x(t), y(t))$ with $\frac{dx}{dt} = \cos t$ and $\frac{dy}{dt} = \sin t$. At time $t = 0$, the object is at position $(6, -1)$. Where is the particle when $t = \pi$?

At $t = \pi$, the object is at $(6,1)$.

$$x(\pi) = x(0) + \int_0^{\pi} \cos t\ dt = 6 + \sin \pi - \sin 0 = 6$$

$$y(\pi) = y(0) + \int_0^{\pi} \sin t\ dt = -1 - \cos \pi + \cos 0 = 1$$

3. An object moving along a curve in the xy-plane has position $p(t) = \langle x(t), y(t)\rangle$. If the acceleration is given by $a(t) = \langle 2, 4e^{2t}\rangle$ with $p(0) = \langle 2,1\rangle$ and the velocity at $t = 0$ is $v(0) = \langle 0,2\rangle$, identify the position and velocity vectors for the object.

$p(t) = \langle t^2 + 2, e^{2t}\rangle$

$v(t) = \langle 2t, 2e^{2t}\rangle$

9.6 SOLVING MOTION PROBLEMS USING PARAMETRIC AND VECTOR-VALUED FUNCTIONS

1. A particle moves according to the parametric equations $x(t) = 2t + 4$ and $y(t) = \sqrt{t} - 1$. Fill in the missing values of the table.

t	1	4	9
x(t)	6	12	22
y(t)	0	1	2
x'(t)	2	2	2
y'(t)	1/2	1/4	1/6

At $t = 1$, choose the statement below that correctly describes the movement of the particle.

I. The particle is moving left and up.

II. The particle is moving left and down.

III. The particle is moving right and up. [The particle is moving right and up at $t = 1$ because both $x'(1)$ and $y'(1)$ are positive.]

IV. The particle is moving right and down.

V. The particle is not moving.

2. A particle's position can be found from $\left(2\cos\left(\frac{t}{2}\right), 4\sin(2t)\right)$ for $t \geq 0$.

Which of the following is the speed of the particle at $t = \pi$?

0 $\qquad$ 3 $\qquad$ 8 $\qquad$ $\sqrt{65}$

$$v(\pi) = \left(-\sin\left(\frac{\pi}{2}\right), 8\cos(2\pi)\right) = (-1, 8)$$

$$|v(\pi)| = \sqrt{(-1)^2 + (8)^2} = \sqrt{65}$$

9.7 DEFINING POLAR COORDINATES AND DIFFERENTIATING IN POLAR FORM

1. For the polar curve $r = 2\cos\theta$, match each expression on the left to the corresponding value on the right.

$x =$	$2(\cos^2\theta - \sin^2\theta)$
$y =$	$2\cos^2\theta$
$\dfrac{dy}{d\theta} =$	$2\cos\theta\sin\theta$
$\dfrac{dx}{d\theta} =$	$-4\cos\theta\sin\theta$

Now use this information to write the equation of the tangent line to the graph at $\theta = \dfrac{\pi}{4}$.

$$y - 1 = 0(x - 1)$$

$$y = 1$$

2. Circle the slope of the line tangent to the polar curve $r = 3\theta^2$ when $\theta = \dfrac{\pi}{2}$.

$-\dfrac{4}{\pi}$ $\qquad \dfrac{\pi}{2}$ $\qquad \dfrac{3\pi^2}{2}$ $\qquad \dfrac{\pi}{4}$

The values of x and y and their derivatives are

$$x = 3\theta^2\cos\theta \qquad \frac{dx}{d\theta} = 6\theta\cos\theta - 3\theta^2\sin\theta$$

$$y = 3\theta^2 \sin\theta \qquad \frac{dy}{d\theta} = 6\theta\sin\theta + 3\theta^2\cos\theta$$

When $\theta = \frac{\pi}{2}$,

$$\frac{dy}{d\theta} = 6\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) + 3\left(\frac{\pi}{2}\right)^2\cos\left(\frac{\pi}{2}\right) = 3\pi(1) + \frac{3\pi^2}{4}(0) = 3\pi$$

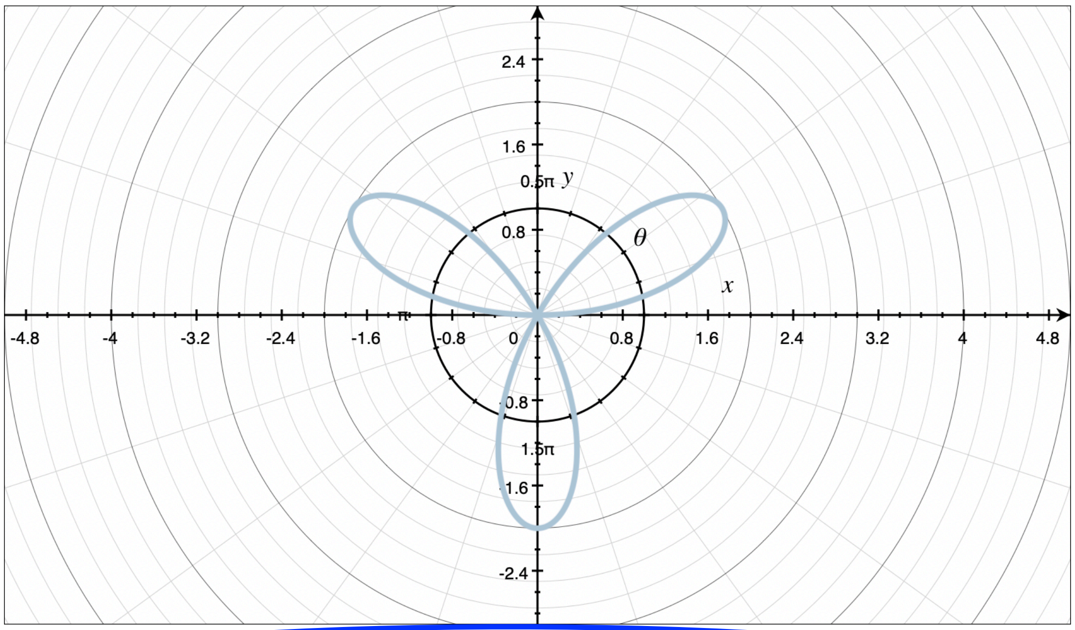
$$\frac{dx}{d\theta} = 6\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) - 3\left(\frac{\pi}{2}\right)^2\sin\left(\frac{\pi}{2}\right) = 0 - \frac{3\pi^2}{4}$$

So $\frac{dy}{dx}$ at $\theta = \frac{\pi}{2}$ is given by

$$\frac{3\pi}{\frac{-3\pi^2}{4}} = \frac{-12\pi}{3\pi^2} = -\frac{4}{\pi}$$

9.8 FIND THE AREA OF A POLAR REGION OR THE AREA BOUNDED BY A SINGLE POLAR CURVE

1. Below is the graph of $r = 2\sin(3\theta)$. Say which of the following three integrals calculates the total area enclosed by the curve.



I. $2\int_0^{\pi} \sin^2(3\theta)\ d\theta$ II. $12\int_0^{\frac{\pi}{6}} \sin^2(3\theta)\ d\theta$ III. $6\int_0^{\frac{\pi}{3}} \sin^2(3\theta)\ d\theta$

All three choices are correct.

2. Which of the following choices give the area of the region enclosed by the polar graph of $r = 3 + 3\sin\theta$.

$\frac{1}{2}\int_0^{\pi} (3 + 3\sin\theta)^2\ d\theta$

$\int_0^{\pi} (3 + 3\sin\theta)^2\ d\theta$

$\frac{1}{2}\int_0^{2\pi} (3 + 3\sin\theta)^2\ d\theta$

$\int_0^{\pi} (3 + 3\sin\theta)^2\ d\theta$

This region would be the graph of a cardioid, which has domain $0 \leq \theta \leq 2\pi$.

3. Which of the following choices gives the area of the inner loop of the curve $r = 3 + 6\cos\theta$?

$\frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (3 + 6\cos\theta)^2\ d\theta$

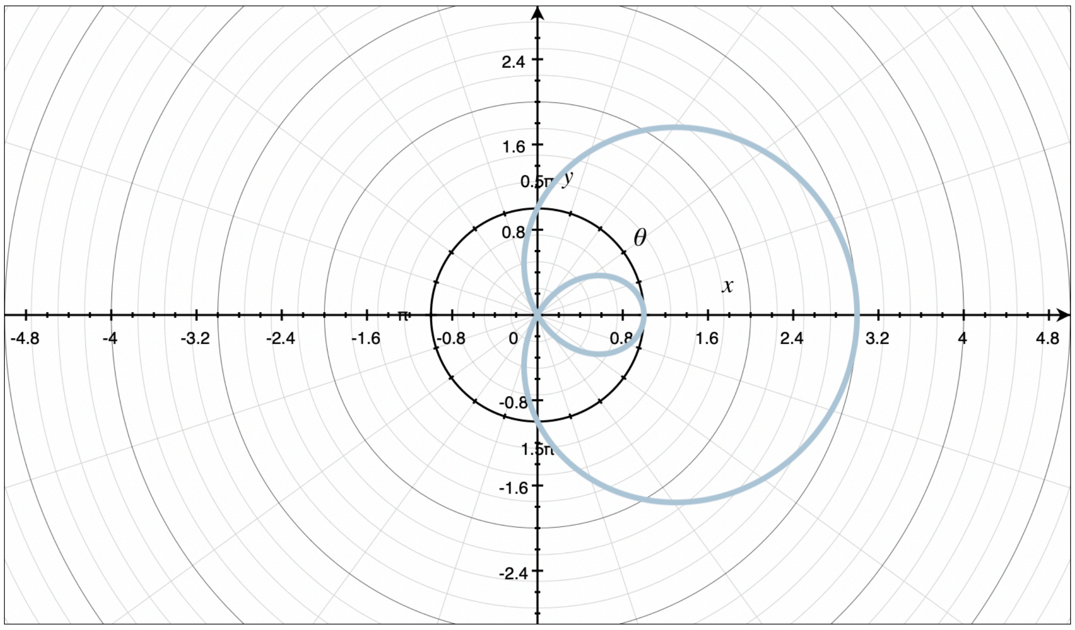
$\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} (3 + 6\cos\theta)^2\ d\theta$

$$\frac{1}{2}\int_0^{2\pi}(3+6\cos\theta)^2\ d\theta$$

$$\frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}(3+6\cos\theta)^2\ d\theta$$

9.9 FINDING THE AREA OF THE REGION BOUNDED BY TWO POLAR CURVES

1. Which of the following four choices will not give the area that falls inside $r=2\cos\theta+1$, but outside the smaller loop.



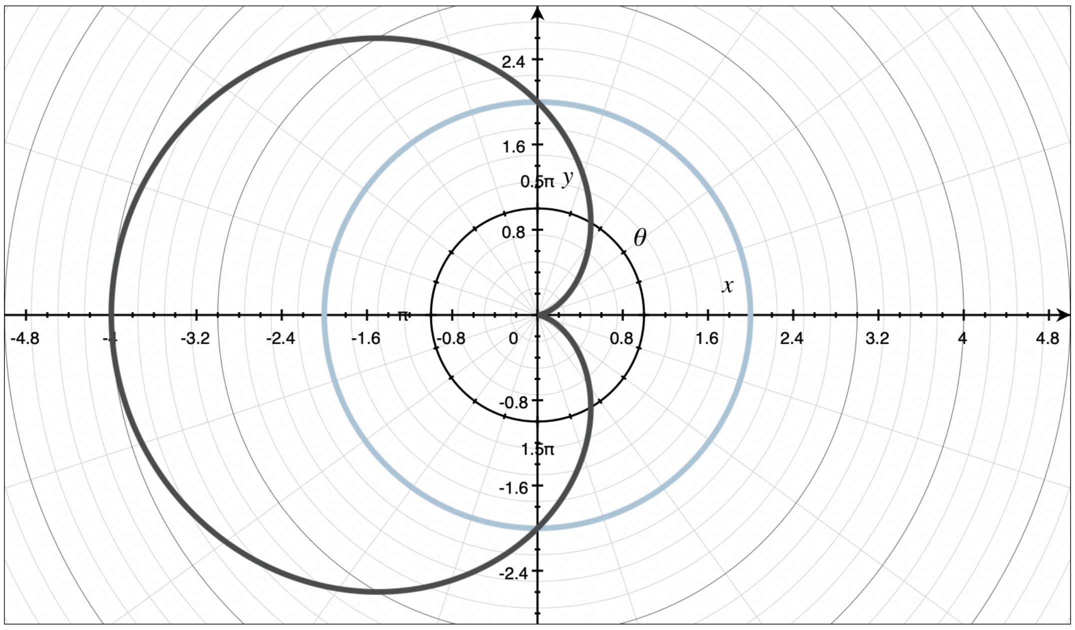
$$\frac{1}{2}\int_0^{2\pi}(2\cos\theta+1)^2\ d\theta-\frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}(2\cos\theta+1)^2\ d\theta$$

$$\frac{1}{2}\int_0^{\frac{2\pi}{3}}(2\cos\theta+1)^2\ d\theta+\frac{1}{2}\int_{\frac{4\pi}{3}}^{2\pi}(2\cos\theta+1)^2\ d\theta$$

$$\int_0^{\frac{2\pi}{3}}(2\cos\theta+1)^2\ d\theta$$

$$\frac{1}{2}\int_{0}^{\frac{4\pi}{3}} (2\cos\theta + 1)^2 \ d\theta$$

2. The graph below shows $r = 2$ and $r = 2 - 2\cos\theta$. Match each of the following integrals with the corresponding statement below.



$$\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4 - (2 - 2\cos\theta)^2 \ d\theta \qquad \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (2 - 2\cos\theta)^2 - 4 \ d\theta$$

$$\frac{1}{2}\int_{0}^{2\pi} 4 \ d\theta \qquad \frac{1}{2}\int_{0}^{2\pi} (2 - 2\cos\theta)^2 \ d\theta$$

The area of the region that is:

inside the circle $\quad \frac{1}{2}\int_{0}^{2\pi} 4 \ d\theta$

inside the cardioid $\quad \frac{1}{2}\int_{0}^{2\pi} (2 - 2\cos\theta)^2 \ d\theta$

inside the circle and outside the cardioid $\quad \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4 - (2 - 2\cos\theta)^2 \ d\theta$

inside the cardioid and outside the circle $\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (2-2\cos\theta)^2 - 4\ d\theta$